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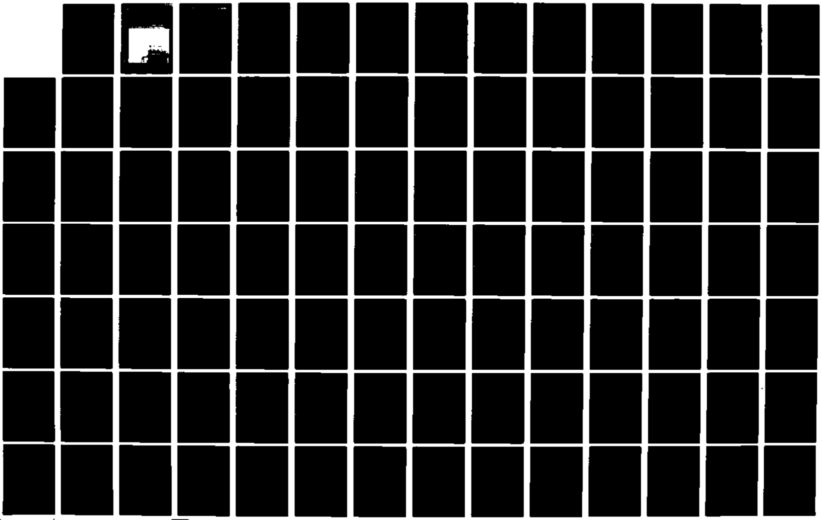

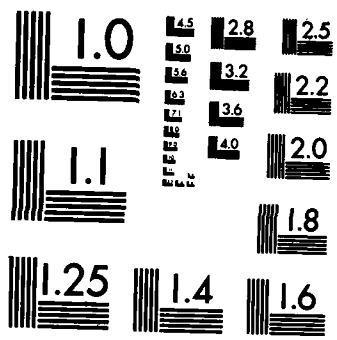

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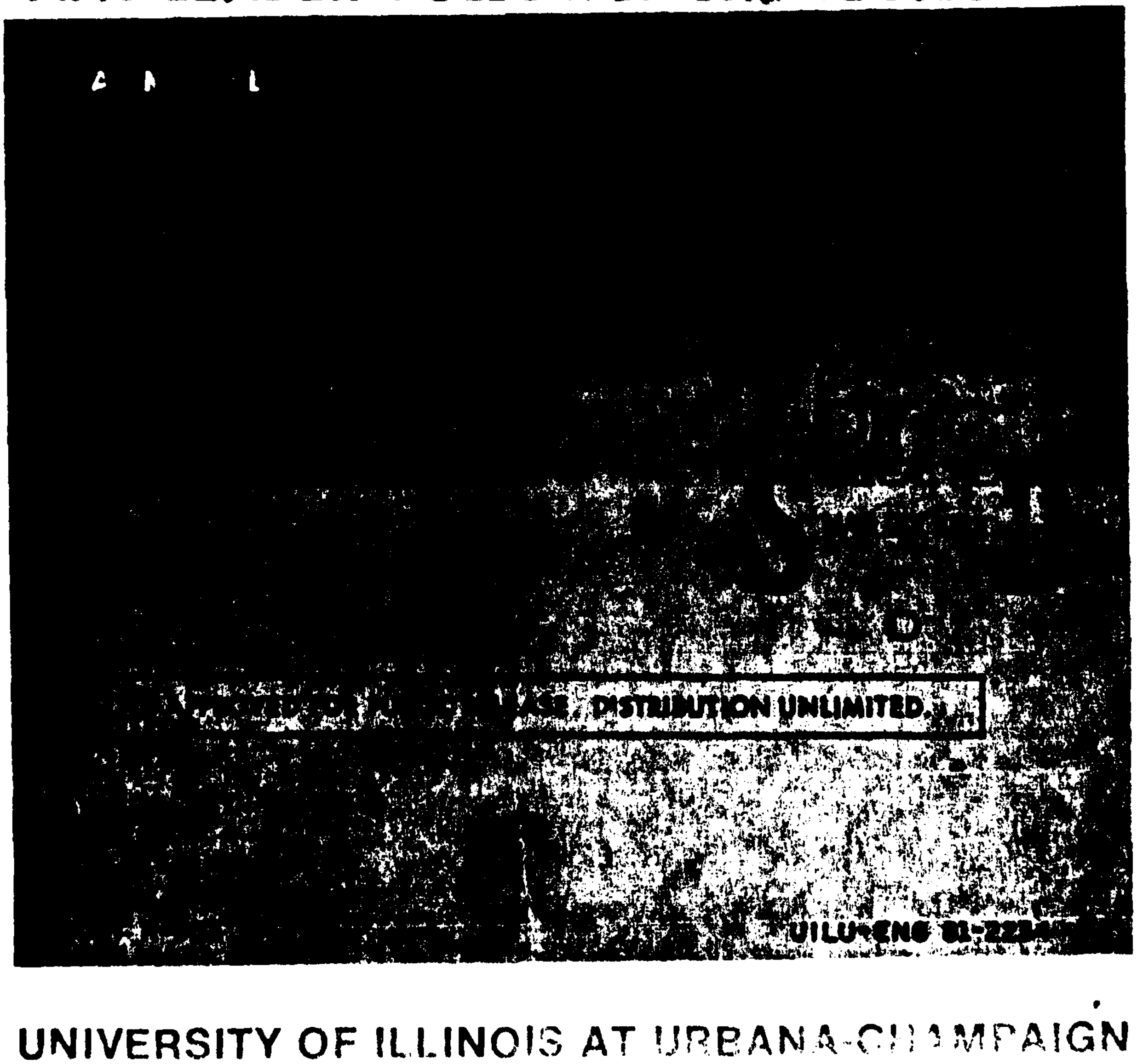
INFORMATION STRUCTURES IN NASH AND LEADER-FOLLOWER STRATEGIES



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The role of information structures in some Nash and Leader-Follower games is examined. By preserving the information structure of the full order singularly perturbed game, while solving the reduced order ones, reduced order near optimal strategies are obtained and well-posedness is shown for two classes of Leader-Follower games. Decision-dependent information structure is employed in both Nash games and optimal coordination problems and two market models of duopoly with this type of information structure are extensively analyzed and examined.

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Mutasim A. Salman

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INFORMATION STRUCTURES IN NASH AND LEADER-FOLLOWER STRATEGIES

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THESIS

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INFORMATION STRUCTURES IN NASH AND LEADER-FOLLOWER STRATEGIES

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CHAPTER 1

INTRODUCTION

1.1. Motivation and Background

In a world complicated by competing multi-national corporations and vast informational data bases stored in high-speed computers, the need for optimizing large scale economic systems has gained considerably in importance. It is thus no surprise that hierarchical engineering solutions to problems of this nature have attracted a great deal of attention from researchers in the past several years. In particular, the theory of games provides formalization of many basic problems in large scale systems, i.e., problems characterized as having several decision makers acting on different sets of information with possibly conflicting goals.

The appearance of the "Theory of Games and Economic Behavior" by J. V. Neumann and O. Morgenstern [1] gave the impetus for research in game theory. Although the importance of the theory of games was initially recognized almost exclusively in economics, its usefulness and recognition as a challenging area of research is established today in mathematics, engineering, economics, sociology, and political science.

A basic feature of the theory is that in solving a game theoretic problem, one is faced not with a conventional minimization (maximization) problem, but with a conceptually different situation altogether. This might be expected, however, since the outcome of a decision maker depends not only on his own actions and chance, but also on complicated interactions with other decision makers. The theory of games can be viewed as a generalization of centralized and decentralized control problems, since most of the questions

posed in control theory can be considered in a game theoretic framework, but their solution is usually more difficult. The generality and unification of several special cases which characterize game theoretic results, help to counterbalance the difficulties encountered in finding them.

Game theory reduces the ingredients in a formalization of the game problem to seven essential elements.

1. The players involved in the game (there may be only two, or possible more players).
2. A description of the interaction both among the players and the players with the system. (In a dynamic setting, a difference or differential equation describes this situation.)
3. The information structure of each player, describing an "information space" which contains the precise information gained or recalled by the player at every stage of the game.
4. The decision space (of alternative course of action) for each player.
5. The admissible strategies of each player, defined as mappings from the information space into the decision space.
6. The objective function of each player.
7. A rational equilibrium solution concept which takes into account the relative power of each player and the hierarchical structure of the decision process.

Two solution concepts which are of particular interest here are the so-called Nash and Leader-Follower (LF) games, (see [30],[3]). A general description of Nash and LF games may be given as follows. Let Γ_1 and Γ_2 be the spaces of admissible strategies for player one and player two,

respectively, with $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Let $J_1(\gamma_1,\gamma_2)$ and $J_2(\gamma_1,\gamma_2)$ be the corresponding cost functions of the two players. A Nash equilibrium solution is defined as follows:

<u>Definition of a Nash equilibrium</u>: (γ_1^o,γ_2^o) is an equilibrium Nash solution if and only if

$$J_1(\gamma_1^o,\gamma_2^o) \le J_1(\gamma_1,\gamma_2^o)$$

and (1.1)

$$J_2(\gamma_1^o,\gamma_2^o) \le J_2(\gamma_1^o,\gamma_2)$$

so a Nash equilibrium solution assumes that if one player minimizes on the basis that the other player's strategy is known and it is at equilibrium then the first player will find his optimal strategy at equilibrium.

To define the LF equilibrium, we need the following definition: The rational reaction set of the follower (player two) to the permissible strategies of the leader (player one), $D^2(\gamma_1)$ is

$$D^2(\gamma_1) = \{\gamma_2^* \in \Gamma_2 \text{ such that } J_2(\gamma_2^*,\gamma_1) \le J_2(\gamma_1,\gamma_2) \text{ for all } \gamma_2 \in \Gamma_2 \text{ and each } \gamma_1\}.$$

<u>Definition of the LF equilibrium</u>: A permissible pair of strategies $(\gamma_1^* \in \Gamma_1,\ \gamma_2^* \in D^2(\gamma_1))$ is said to be in global LF equilibrium, with player one as a leader, if

$$J_1(\gamma_1^*,\gamma_2^*) \le J_1(\gamma_1,\gamma_2)$$

for all pairs

$$\{\gamma_1 \in \Gamma_1,\ \ \gamma_2 \in D^2(\gamma_1)\}.$$

Nash game describes a situation of conflict where the two players do not trust each other and do not cooperate, but each player assumes that the other

one will act in a rational way. The two players are assumed to declare, but not necessarily apply, their strategies simultaneously, and both players know Γ_1, Γ_2, J_1, and J_2. The LF game formalizes the situation where the follower tries to minimize J_2 for a given choice of $\gamma_1 \in \Gamma_1$ by the leader. The leader, who wants to minimize J_1, knowing the rational reaction of the follower and having the power to declare his strategy first, wishes to announce a strategy γ_1^* which achieves the minimum possible J_1. The leader must know Γ_1, Γ_2, J_1, and J_2, but the follower knows the strategy γ_1^* (not the strategy value), Γ_2 and J_2.

The LF solution concept was first introduced by von Stackelberg [3] within the context of economic competition. It was generalized to the dynamic game case by Chen, Cruz, and Simaan ([4],[5],[6]). There are different types of LF strategies, (a) open-loop strategies, (b) feedback strategies, and (c) closed-loop strategies. References [5] and [6] provide discussions of these various types; however, we will confine our attention primarily to the closed-loop type. It was thought for a long time that the solution of closed-loop strategies would be impossible to obtain, but recently several successful attempts have been made in this direction, see [16], [18], and [26]. In [18] and [26] a new and important class of LF games of the closed-loop type has been defined and developed. In this class the leader achieves his global optimal payoff. In other words, the leader is able to induce the follower to play with him as a team, even though the follower optimizes his objective function.

The information gained or recalled by each decision maker (DM) at every stage of the game (the information structure), is crucial to the solution of

the game. For a given information structure, each DM tries to find his optimal strategy (where optimality is defined according to the solution concept adopted by the DM). A different information structure will generally require a different optimal strategy, and hence may result in a different payoff. This leads to a natural way for comparing two information structures in terms of the maximum payoff that can be achieved through their use. For a detailed discussion of information structure and its value, the reader is referred to [44] and [48]. In problems involving a single decision maker, the more information the DM has, the better off he is; but, for problems with many decision makers this is not generally true, as was shown in [54,55,57] within the context of Nash games.

1.2. Contribution and Outline of the Thesis

In this section we will outline the results of our work, relate them to other existing ones, and point out our contributions.

In this thesis we consider two important aspects of information structures in Nash and LF games. These two aspects are preservation of information structure and decision-dependent information structure. By preserving the information structure of the full order game, while solving the reduced order ones, we obtain reduced order solutions which are equivalent to the full order ones. But by using decision-dependent information structure, we obtain solutions which are usually different from but more desirable than the solutions obtained by using normal information structures; in particular, in LF games, by using decision dependent information structure, the leader

can, under certain conditions, achieve any feasible solution he desires for the game.

In Part One of the thesis, we show that by preserving the information structure of the full-order singularly-perturbed LF games, while solving the reduced-order ones, reduced-order and near-optimal strategies are obtained and well-posedness is achieved.

Singular perturbation technique is used to decrease the order of the system, and hence reduce computation and alleviate the numerical "stiffness" in the problem. Alternatively we can regard the model reduction as a simplification in desiring to obtain approximate strategies which are asymptotically optimal but which involve significant reduction in computation. A fundamental question is whether the resulting reduced optimization function is, in the limit, equal to the full order optimization function for each player as μ tends to zero. In linear quadratic control problems the reduced and the full order optimization are equal in the limit as μ tends to zero, i.e. the usual singular perturbation procedure is well-posed [9]. However, in games this is not generally true, as was shown by a counterexample in [10]. When the usual singular perturbation procedure leads to an ill-posed solution, it is desirable to seek a modified procedure which is well-posed. Such well-posed order reduction have been obtained for linear quadratic Nash games [10]. Cases for which the usual singular perturbation leads to a well-posed solution have been reported in [22].

In Chapter 2 the well-posedness of linear closed-loop LF strategy is considered. When the space of closed-loop LF strategies is constrained to be a linear function of the state variables, it was found [8] that such linear strategies do not exist because some gain matrices depend on the

initial conditions. But by assuming that the initial conditions are randomly distributed and averaging the performance indices over these initial conditions, linear closed-loop LF strategies were obtained. In [11] a linear closed-loop LF strategy as described in [8], was considered, and it was shown that if we restrict the space of strategies to be taken from the slow variable only, we obtain a well-posed formulation. In [56] and in this chapter, we consider the linear closed-loop LF strategy when both the slow and the fast are available for measurement. This information structure is different from the one in [11]. We introduce a method by which we can find strategies using reduced order systems such that if we apply these strategies to the full order system, the resulting cost functions will have the same limits as the cost function for the same full order systems if the full order optimal strategies are applied. Preserving the information structure of the full order problem is the basic feature of our procedure. In Chapter 3 we consider team LF games for singularly perturbed systems. We design a well-posed method to obtain reduced order near optimal strategies. In this method, we solve two subgames, one for the fast modes and the other for hybrid slow modes (hybrid because the fast gain is imbedded), under the constraint that the information structure of the full order problem is preserved. We also show that the sufficient conditions for existence of a team LF solution for the reduced order games is equivalent to those of the full order one in the limit as μ tends to zero.

In Part Two of this thesis, decision-dependent information structure (DDIS) is employed in some classes of Nash and LF games. We introduce and analyze two new models of duopoly with this type of DDIS and we give sufficient conditions for the existence of the solution of two classes of

stochastic team-LF games with DDIS. In Chapter 4 we consider Nash games with DDIS and introduce a market model of duopoly. Although the concept of DDIS in Nash games is not new (see [38] for example) our analysis and approach are different. In Section 4.2 we formally define the equilibrium Nash solution with the two types of information structure and we give two examples which clarify the basic ideas in this section. In Section 4.3 we consider a general static market model of duopoly and derive the necessary conditions for the supply adjustment controls of both firms to be optimal in the Nash sense for the two types of information structure. Then we analyze, in detail, the special case of a linear market demand relation and quadratic cost function. In Section 4.4 we generalize the concept of DDIS to multistage dynamic games, and we give sufficient conditions for existence of the Nash equilibrium solution with DDIS for the discrete linear quadratic problem.

In Chapter 5 we consider DDIS as the incentive mechanism (we refer the interested reader to [40],[41],[42] for discussions on the incentive problem), which is used by the leader to induce the Nash followers, in a LF game, to behave as members of a team with the leader's objective as the objective of the team, and develop a static market model of duopoly with the government as the market coordinator. In Section 5.2, the incentive problem is formalized as (n+1)-person LF game with one leader and n-Nash followers. The leader desires to force the followers to optimize his (the leader's) objective function, even though each one optimizes his own objective function. In Section 5.3 the incentive mechanism of the organization is formulated by incorporating the decisions of the followers in the strategy of the leader. By employing such forms of strategies, the leader can force the followers to behave as members of a team, with their composite objective function contained

in the objective function of the team. In Section 5.4, we consider a general static market model of duopoly where the government interfers in the market. We show that the government can always succeed in enforcing the two duopolists to cooperate and achieve the Pareto-optimal solution. Then we analyze in detail the case of a linear market demand relation and a quadratic cost function. We obtain analytic solutions for the optimal strategies of the two Nash duopolists and the government. We show that in the limit as the unit cost of the govenment control tends to infinity, the enforced cooperative optimal controls and profits tend to the voluntary cooperative ones. Finally, we discuss the general properties in terms of marginal cost, price, and the consumers' welfare in the context of this problem. In the last chapter we deal with two stochastic static LF problems, where the leader can, by using DDIS, achieve the team solution. Each player has a quadratic cost function and part of his information is a linear function of Gaussian random variables.

Answers to many problems in the area of stochastic control with classical information structure (in classical information structure, all actions taken at the same time are based on the same information, and any information available at time t will still be available at time $t' > t$) are known; in particular, the problem of linear quadratic Gaussian (LQG) is completely solved. But, stochastic control problems with nonclassical information structure are more difficult. These problems are usually viewed in the context of team theory. The most important theorem in team theory is Radner's theorem on teams with static information structure (see [44] for the statement of the theorem). By using the concept of nesting, this theorem was extended, in [43], to solve problems of teams with dynamic information structure. Later, team decision making problems were generalized to stochastic game

theoretic framework. In [49] a two person stochastic Nash game with static information structure was considered, and it was shown that under some sufficient conditions the linear quadratic Gaussian Nash game admits a unique equilibrium solution which is linear in the observation of each decision maker. In [52] a linear quadratic Gaussian two person LF team game with decision-dependent information structure was considered and completely investigated.

In the first section of Chapter 6 we consider the problem of a 3-person stochastic optimal coordination, where the coordinator desires to induce the two noncooperative (in Nash sense) players to minimize his (the leader) cost function, even though each player minimizes his own cost function. The coordinator's cost function is a convex combination of the cost functions of the Nash players. The information structure of the game is dynamic and nested since the coordinator knows whatever the fohlower knows, and his (the coordinator's) strategy depends on the decisions of the followers. In the second section, we investigate a 2-person LF game in which the leader does not completely detect the decision variable of the follower. The satisfaction of the condition of complete detectability of the action of the follower is necessary for the leader to be able to obtain his global optimal solution. The case in which the leader does not completely detect the action variable of the follower was investigated in a deterministic setting by Basar [47]. He gave a general procedure by which the leader can achieve a new tight lower bound for his objective function. In this section we solve the same problem. but in a stochastic setting, and using a different procedure. We define a new modified team problem after taking into account the optimal response of the undetected action of the follower. We find that the leader can, under a certain condition, achieve this new tight lower bound.

Finally, in Chapter 7 we state some conclusions and we outline directions for future research.

In summary, the basic contributions of this thesis are

1. Introduction of two well-posed procedures to obtain reduced order and near optimal strategies for two classes of singularly perturbed LF games. The key feature of these two procedures is the preservation of the information structure of the full order game, while solving the reduced order ones.
2. Introduction of two new market models of duopoly with DDIS and a detailed analysis of these models.
3. Determination of sufficient conditions for the existence of solutions for both the stochastic optimal coordination problem with DDIS and the stochastic LF team problem with partial decision-dependent information structure.

PART ONE

INFORMATION STRUCTURE IN SINGULARLY PERTURBED GAMES

CHAPTER 2

WELL-POSEDNESS OF LINEAR CLOSED-LOOP LEADER-FOLLOWER STRATEGIES

2.1. Introduction

In this chapter, we describe a well-posed procedure to obtain reduced order and near optimal strategies for singularly perturbed linear closed LF games. In Section 2.2, we formulate the problem and give the necessary conditions for existence of a linear closed loop LF equilibrium solution for the full order system. In Section 2.3, the follower minimizes the fast part of his objective function under the condition that the fast subsystem and the fast part of the leader's strategy are given. In Section 2.4, the follower minimizes his slow part of his objective function under the conditions that the slow subsystem is given and that the follower's and leader's strategies have the same information structure as their corresponding strategies in the full order game. In Section 2.5, the leader minimizes his slow part of his objective function under the conditions that the slow subsystem is given and that the follower applied the above procedure to find his rational reaction. In Section 2.6, we find that if we apply the reduced order optimal gains to the full order system, the resulting strategies and cost functions will have the same limits as the strategies and the cost functions for the same full order system if the full order optimal gains are applied.

2.2. Formulation of the Problem

Let us consider the singularly perturbed system:

$$\dot{x} = A_{11}x + A_{12}z + B_{11}u_1 + B_{12}u_2; \qquad x(0) = x_0$$

$$\varepsilon\dot{z} = A_{21}x + A_{22}z + B_{21}u_1 + B_{22}u_2; \qquad z(0) = z_0$$

where $x \in R^{n_1}$, $z \in R^{n_2}$; $u_i \in R^{m_i}$ and ε is a small positive parameter. Assume that the cost function associated with player i is

$$\bar{J}_i = \underset{y_o}{E}\left[\tfrac{1}{2}\int_o^{\infty}\{y'Q_i y + u_i'R_{ii}u_i + u_j'R_{ij}u_j\}dt\right]$$

where

$$y = \begin{bmatrix} x \\ z \end{bmatrix} \quad ; \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix}$$

R_{ii}, R_{ij} are symmetric, positive definite matrices, $E(y_o) = 0$; $E(y_o y_o') = I$ where I is the identity matrix.

Let player 2 be the leader and player 1 be the follower. A closed-loop linear Stackelberg strategy was considered by Medanic [8]. In his paper the controls were assumed to be of the form

$$u_1 = -F_1 y \ , \quad u_2 = -F_2 y.$$

and F_2, the gain of the leader is found by solving the following equations.

$$A_c'M_1 + M_1A_c + M_1S_{11}M_1 + F_2'R_{12}F_2 + Q_1 = 0 \tag{2.1a}$$

$$A_c'M_2 + M_2A_c + M_1S_{21}M_1 + F_2'R_{22}F_2 + Q_2 = 0 \tag{2.1b}$$

$$N_1A_c' + A_cN_1 - S_{11}M_2N_2 - N_2M_2S_{11} + S_{21}M_1N_2 + N_2M_1S_{21} = 0 \tag{2.1c}$$

$$N_2A_c' + A_cN_2 + I = 0 \tag{2.1d}$$

$$R_{12}F_2N_1 + R_{22}F_2N_2 - B_2'(M_1N_1 + M_2N_2) = 0 \tag{2.1e}$$

where

$$F_1 = R_{11}^{-1}B_1'M_1$$

$$A_c = A - S_{11}M_1 - B_2F_2$$

$$S_{ij} = B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j'$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\mu} & \frac{A_{22}}{\mu} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1i} \\ \frac{B_{2i}}{\mu} \end{bmatrix}, \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q'_{i2} & Q_{i3} \end{bmatrix}$$

In general by letting $\mu \to 0$ in the full order system we change the meaning of the vector z from a state variable to a variable which depends on x. So if we solve the resulting slow optimization problem, we will have a change in information structure. To avoid this change in information structure, we solve the problem as an output feedback problem, where we constrain the feedback to be taken from x and z. This is clearly shown in Section 2.4.

In the following sections we will show a procedure to get a well-posed solution of the problem depending on reduced order systems while both x and z are available for measurements for both players. Let

$$u_{1ap} = -L_{11}x - L_{12}z.$$

$$u_{2ap} = -L_{21}x - L_{22}z.$$

The follower will find L_{12} by minimizing the fast part of his optimization function while the fast part of the system is given, and he will find L_{11} by minimizing his modified slow optimization function. The leader will find his gains L_{21} and L_{22} by minimizing his slow part of the optimization function under the constraints that the follower applies the above procedure and the slow part of the system is given.

2.3. The Fast Optimization Problem for the Follower

The follower can find the gain L_{12} by minimizing the fast part of his performance index which is

$$\bar{J}_{1f} = \underset{z_f(0)}{E}\left[\frac{1}{2}\int_0^\infty (z_f' Q_{13} z_f + u_{1f}' R_{11} u_{1f} + u_{2f}' R_{12} u_{2f})dt\right]$$

given that

$$\mu \frac{dz_f}{dt} = A_{22} z_f + B_{21} u_{1f} + B_{22} u_{2f}$$

and

$$u_{1f} = -L_{12} z_f$$

$$u_{2f} = -L_{22} z_f.$$

Substituting for u_{1f}, u_{2f}, we get

$$\bar{J}_{1f} = \underset{z_f(0)}{E}\left[\frac{1}{2}\int_0^\infty (z_f' Q_{13} z_f + L_{12}' R_{11} L_{12} + L_{22}' R_{12} L_{22})dt\right]$$

$$\mu \dot{z}_f = (A_{22} - B_{21} L_{12} - B_{22} L_{22}) z_f = \hat{A}_{22} z_f.$$

Solving the problem, we get the following necessary conditions

$$L_{12} = R_{11}^{-1} B_{21}' K_{13} \tag{2.2}$$

$$\hat{A}_{22}' K_{13} + K_{13} \hat{A}_{22} + K_{13} \bar{S}_{13} K_{13} + L_{22}' R_{12} L_{22} + Q_{13} = 0 \tag{2.3}$$

where

$$\bar{S}_{13} = B_{21} R_{11}^{-1} B_{21}'.$$

2.4. Hybrid-Slow Optimization Problem for the Follower

The follower can find L_{11} by the following procedure. Letting $\mu \to 0$ in the system considered we obtain

$$\dot{x}_s = A_{11}x_s + A_{12}z_s + B_{11}u_{1s} + B_{12}u_{2s}$$

$$0 = A_{21}x_s + A_{22}z_s + B_{21}u_{1s} + B_{22}u_{2s}$$

and if we constrain the controls to be of the form

$$u_{1s} = -L_{11}x_s - L_{12}z_s$$

$$u_{2s} = -L_{21}x_s - L_{22}z_s$$

and substitute for u_{1s}, u_{2s}, we obtain

$$z_s = -(A_{22}-B_{21}L_{12} - B_{22}L_{22})^{-1}(A_{21}-B_{21}L_{11}-B_{22}L_{21})x_s.$$

Assuming that $(A_{22}-B_{21}L_{12}-B_{22}L_{22})$ is nonsingular and substituting for u_{1s}, u_{2s}, z_s in the differential equation, we obtain

$$\dot{x}_s = [A_{11}-B_{11}L_{11}-B_{12}L_{21}-(A_{12}-B_{11}L_{12}-B_{12}L_{22})(A_{22}-B_{21}L_{12}-B_{22}L_{22})^{-1}$$
$$-(A_{21}-B_{21}L_{11}-B_{22}L_{21})]x_s$$

or

$$\dot{x}_s = A_o x_s \ , \quad x_s(t) = \Phi(t,0)x(0)$$

where

$$A_0 = \hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}$$

$$\hat{A}_{11} = A_{11} - B_{11}L_{11} - B_{12}L_{21}, \quad \hat{A}_{12} = A_{12} - B_{11}L_{12} - B_{12}L_{22}$$

$$\hat{A}_{21} = A_{21} - B_{21}L_{11} - B_{22}L_{21}, \quad \hat{A}_{22} = A_{22} - B_{21}L_{12} - B_{22}L_{22}.$$

Substituting for u_{1s}, u_{2s}, z_s in the optimization function of the follower, we obtain

$$\bar{J}_{1s} = \underset{x_o}{E}\Big[\frac{1}{2}\int_0^\infty x_o'\{\phi'(t,o)[Q_{11}+L_{11}'R_{11}L_{11}+L_{21}'R_{12}L_{21} - (Q_{12}+L_{11}'R_{11}L_{12}+L_{21}'R_{12}L_{22})$$

$$\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'(Q_{12}'+L_{12}'R_{11}L_{11}+L_{22}'R_{12}L_{21}) + \hat{A}_{21}'(\hat{A}_{22}^{-1})'$$

$$(Q_{13}+L_{12}'R_{11}L_{12}+L_{22}'R_{12}L_{22})\hat{A}_{22}^{-1}\hat{A}_{21}]\phi(t,o)\}x_o dt\Big].$$

Applying the same procedure as in the output regulator problem [12], while using the assumption that $E(x_o x_o') = I$, we obtain

$$\bar{J}_{1s}(L_{11},L_{12}) = \tfrac{1}{2}\mathrm{trace}\int_0^\infty \phi'(t,0)[\tilde{Q}_{11} - \tilde{Q}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{12}' +$$

$$\hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{13}\hat{A}_{22}^{-1}\hat{A}_{21}]\phi(t,0)dt$$

where

$$\tilde{Q}_{11} = Q_{11} + L_{11}'R_{11}L_{11} + L_{21}'R_{12}L_{21}$$

$$\tilde{Q}_{12} = Q_{12} + L_{11}'R_{11}L_{12} + L_{21}'R_{12}L_{22}$$

$$\tilde{Q}_{13} = Q_{13} + L_{12}'R_{11}L_{12} + L_{22}'R_{12}L_{22}.$$

Finding $\dfrac{\partial \bar{J}_{1s}}{\partial L_{11}}$ and setting it equal to zero we obtain

$$R_{11}L_{11} - R_{11}L_{12}\hat{A}_{22}^{-1}\hat{A}_{21} + B_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{12}' - B_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{13}\hat{A}_{22}^{-1}\hat{A}_{21} = \tilde{B}_1'K_{11} \quad (2.4)$$

where

$$\tilde{B}_1' = B_{11}' - B_{21}'(\hat{A}_{22}^{-1})'(\hat{A}_{12})'$$

and K_{11} is the solution of

$$K_{11}A_0 + A_0'K_{11} + \tilde{Q}_{11} - \tilde{Q}_{12}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{12}' + \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{13}\hat{A}_{22}^{-1}\hat{A}_{21} = 0. \tag{2.5}$$

Substituting for L_{12} obtained from equation (2.2), (2.3) in equation (2.4), (2.5), we can find L_{11}.

Comments:

(1) Finding $\frac{\partial \bar{J}_{1s}}{\partial L_{12}}$ and letting it equal to zero will lead to the same equations as (2.4), (2.5). This is due to the fact that $z_s(t)$ is a linear function of $x_s(t)$.

(2) If we constrain $u_{1s} = -L_{11}x_s$, and apply the same procedure, the formulation is ill-posed.

2.5. The Leader Problem

Before describing how the leader can find L_{21}, L_{22}, it is advantageous to change the form of equations (2.4) and (2.5) by using equations (2.2) and (2.3), and by letting L_{11} to be of the following form

$$L_{11} = R_{11}^{-1}(B_{11}'K_{11} + B_{21}'K_{12}')$$

Then after some straightforward but lengthy algebra equation (2.4) leads to

$$K_{12} = -[Q_{12} + L_{21}'R_{12}L_{22} + K_{11}A_{12} - K_{11}B_{12}L_{22} + \tilde{A}_{21}'K_{13}]\hat{A}_{22}^{-1} \tag{2.6}$$

where

$$\tilde{A}_{21} = A_{21} - \bar{S}_{12}'K_{11} - B_{22}L_{21}$$

$$\bar{S}_{12} = B_{11}R_{11}^{-1}B_{21}' , \quad \bar{S}_{11} = B_{11}R_{11}^{-1}B_{11}'$$

and equation (2.5) becomes

$$K_{11}A_{11} + A_{11}'K_{11} - K_{11}\bar{S}_{11}K_{11} - K_{11}B_{12}L_{21} - L_{21}'B_{12}'K_{11} + Q_{11} - K_{12}\bar{S}_{13}K_{12}' +$$

$$K_{12}\tilde{A}_{21} + \tilde{A}_{21}'K_{12}' + L_{21}'R_{12}L_{21} = 0. \qquad (2.7)$$

By substituting for u_{1s}, u_{2s}, z_s in the slow part of the leader's optimization function, we obtain

$$\bar{J}_{2s} = \frac{1}{2}\underset{x_o}{E}[\int_0^\infty x_s'[\tilde{Q}_{21} - \tilde{Q}_{22}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{22}' + \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{23}\hat{A}_{22}^{-1}\hat{A}_{21}]x_s(t)dt]$$

where

$$\tilde{Q}_{21} = Q_{21} + L_{21}'R_{22}L_{22} + K_{12}\bar{S}_{22}'K_{11} + K_{12}\bar{S}_{23}K_{12}' + K_{11}\bar{S}_{21}K_{11} + K_{11}\bar{S}_{22}K_{12}'$$

$$\tilde{Q}_{22} = Q_{22} + L_{21}'R_{22}L_{22} + K_{12}\bar{S}_{23}K_{13} + K_{11}\bar{S}_{22}K_{13}$$

$$\tilde{Q}_{23} = Q_{23} + L_{22}'R_{22}L_{22} + K_{13}\bar{S}_{23}K_{13}$$

$$\bar{S}_{21} = B_{11}R_{11}^{-1}R_{21}R_{11}^{-1}B_{11}', \quad \bar{S}_{22} = B_{11}R_{11}^{-1}R_{21}R_{11}^{-1}B_{21}', \quad \bar{S}_{23} = B_{21}R_{11}^{-1}R_{21}R_{11}^{-1}B_{21}'$$

Let $\bar{J}_{2s} = \underset{x(0)}{E}[\frac{1}{2}x'(0)K_2x(0)]$, where K_2 satisfies

$$A_0'K_2 + K_2A_0 + \tilde{Q}_{21} - \tilde{Q}_{22}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{22}' + \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{23}\hat{A}_{22}^{-1}\hat{A}_{21} = 0. \qquad (2.8)$$

So the leader has to minimize $\bar{J}_{2s}$ under the following constraints:

$$\hat{A}_{22}'K_{13} + K_{13}\hat{A}_{22} + K_{13}\bar{S}_{13}K_{13} + L_{22}'R_{12}L_{22} + Q_{13} = 0$$

$$K_{11}A_{11} + A_{11}'K_{11} - K_{11}\bar{S}_{11}K_{11} - K_{11}B_{12}L_{21} - L_{21}'B_{12}'K_{11} + Q_{11} - K_{12}\bar{S}_{13}K_{12}' +$$

$$K_{12}\tilde{A}_{21} + \tilde{A}_{21}'K_{12}' + L_{21}'R_{12}L_{21} = 0$$

$$A_0'K_2 + K_2A_0 + \tilde{Q}_{21} - \tilde{Q}_{22}\hat{A}_{22}^{-1}\hat{A}_{21} - \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{22}' + \hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{23}\hat{A}_{22}^{-1}\hat{A}_{21} = 0$$

where

$$K_{12} = -[Q_{12} + L_{21}'R_{12}L_{22} + K_{11}A_{12} - K_{11}B_{12}L_{22} + \tilde{A}_{21}'K_{13}]\hat{A}_{22}^{-1}$$

The reader is referred to Appendix A for the derivation of the necessary conditions for the leader's minimization problem.

2.6. Full Order Problem

In equation (2.1) we assume

$$M_i = \begin{bmatrix} M_{i1} & \mu M_{i2} \\ \mu M_{i2}' & \mu M_{i3} \end{bmatrix} \qquad N_i = \begin{bmatrix} N_{i1} & N_{i2} \\ N_{i2}' & N_{i3} \end{bmatrix}$$

Substituting for M_1 in equation (2.1a) and letting $\mu \to 0$, we obtain the following

$$A_{c11}'M_{11}(0) + M_{11}(0)A_{c11} + A_{c21}'M_{12}'(0) + M_{12}(0)A_{c21} + M_{11}(0)(\bar{S}_{11}M_{11}(0) + \bar{S}_{12}M_{12}'(0))$$
$$+M_{12}(0)(\bar{S}_{12}'M_{11}(0) + \bar{S}_{13}M_{12}'(0)) + F_{21}'(0)R_{12}F_{21}(0) + Q_{11} = 0 \qquad (2.9)$$

$$M_{12}(0) = -[Q_{12} + F_{21}'(0)R_{12}F_{21}(0) + (A_{21} - \bar{S}_{12}M_{11}(0) - B_{22}F_{21}(0))'M_{13}(0) + M_{11}A_{12}$$
$$-M_{11}(0)B_{12}F_{22}(0)]A_{c22}^{-1} \qquad (2.10)$$

$$A_{c22}'M_{13}(0) + M_{13}(0)A_{c22} + M_{13}(0)\bar{S}_{13}M_{13}(0) + F_{22}'(0)R_{12}F_{22}(0) + Q_{13} = 0 \qquad (2.11)$$

where

$$A_{c11} = A_{11} - \bar{S}_{11}M_{11}(0) - \bar{S}_{12}M_{12}'(0) - B_{12}F_{21}(0)$$

$$A_{c12} = A_{12} - \bar{S}_{12}M_{13}(0) - B_{12}F_{22}(0)$$

$$A_{c21} = A_{21} - \bar{S}'_{12} M_{11}(0) - \bar{S}_{13} M'_{12}(0) - B_{22} F_{21}(0)$$

$$A_{c22} = A_{22} - \bar{S}_{13} M_{13}(0) - B_{22} F_{22}(0)$$

assuming that A_{c22} is nonsingular.

It is noticed that equations (2.3), (2.6) and (2.7) are identical to equations (2.11), (2.10) and (2.9) respectively where $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace K_{11}, K_{12}, K_{13}, L_{21}, L_{22}.

Substituting for M_2 in equation (2.1b) and letting $\mu \to 0$, we obtain

$$M_{21}(0)A_{c11} + A'_{c11}M_{21}(0) + M_{22}(0)A_{c21} + A'_{c22}M'_{22}(0) + Q_{21} + F'_{21}(0)R_{22}F_{21}(0) + M_{11}(0)$$
$$(\bar{S}_{21}M_{11}(0) + \bar{S}_{22}M'_{12}(0)) + M_{12}(0)(\bar{S}'_{22}M_{11}(0) + \bar{S}_{23}M'_{12}(0)) = 0 \qquad (2.12)$$

$$M_{21}(0)A_{c12} + M_{22}(0)A_{c22} + A_{c21}M_{23}(0) + M_{11}(0)\bar{S}_{22}M_{13}(0) + M_{12}(0)\bar{S}_{23}M_{13}(0) +$$
$$F'_{21}(0)R_{22}F_{22}(0) + Q_{22} = 0 \qquad (2.13)$$

$$A'_{c22}M_{23}(0) + M_{23}(0)A_{c22} + M_{13}(0)\bar{S}_{23}M_{13}(0) + F'_{22}(0)R_{22}F_{22}(0) + Q_{23} = 0. \qquad (2.14)$$

From (2.13) we have

$$M_{22} = -[A'_{c21}M_{23}(0) + M_{21}(0)A_{c12} + M_{11}(0)S_{22}M_{13}(0) + M_{12}(0)S_{23}M_{13}(0) + F'_{21}(0)R_{22}$$
$$F_{22}(0) + Q_{22}]A^{-1}_{c22}.$$

Substituting for M_{22} in (2.12) and using equation (2.14) we obtain an equation identical to (2.8), where $M_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace K_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22} respectively.

Decomposing equation (2.1d), and letting $\mu \to 0$ we obtain

$$A_{c11}N_{21}(0) + N_{21}(0)A'_{c11} + A_{c12}N'_{22}(0) + N_{22}(0)A'_{c12} + I = 0 \tag{2.15}$$

$$N_{21}(0)A'_{c21} + N_{22}(0)A'_{c22} = 0 \tag{2.16}$$

$$A_{c21}N_{22} + N'_{22}A'_{c21} + A_{c22}N_{23} + N_{23}A'_{c22} = 0. \tag{2.17}$$

From (2.16), we get $N_{22}(0) = -N_{21}(0)A'_{c21}(A^{-1}_{c22})' = N_{21}(0)V'$.

From (2.17), we get $N_{23}(0) = VN_{21}(0)V'$

where $\quad V = -A^{-1}_{c22}A_{c21}$.

Substituting for N_{22} in (2.15), we obtain

$$(A_{c11}-A_{c12}A^{-1}_{c22}A_{c21})N_{21}(0) + N_{21}(0)(A_{c11}-A_{c12}A^{-1}_{c22}A_{c21})' + I = 0$$

which is identical to (A.2) where $N_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace P_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22} respectively.

After decomposing (2.1c), and letting $\mu \to 0$, we obtain

$$\begin{aligned}&[A_{c11}N_{11}(0) + A_{c12}N'_{12}(0) - (\bar{S}_{11}M_{21}(0) + \bar{S}_{12}M'_{22}(0))N_{21}(0) - \bar{S}_{12}M_{23}(0)N'_{22}(0) + \\ &(\bar{S}_{21}M_{11}(0) + \bar{S}_{22}\dot{M}'_{12}(0))N_{21}(0) + \bar{S}_{22}M_{13}(0)N'_{22}(0)] + [N_{11}(0)A'_{c11} + N_{12}(0)A'_{c12} \\ &-N_{21}(0)(M_{21}(0)\bar{S}_{11} + M_{22}(0)\bar{S}'_{12}) - N_{22}(0)M_{23}(0)\bar{S}'_{12} + N_{21}(0)(M_{11}(0)\bar{S}_{21} + M_{12}(0)\bar{S}'_{22}) \\ &\quad + N_{22}(0)M_{13}(0)\bar{S}'_{22}] = 0\end{aligned} \tag{2.18}$$

$$\begin{aligned}&N_{11}(0)A'_{c21} + N_{12}(0)A'_{c22} + N_{21}(0)(M_{21}(0)\bar{S}_{21} + M_{22}(0)\bar{S}_{13}) - N_{22}(0)M_{23}(0)\bar{S}_{13} \\ &\quad + N_{21}(0)(M_{11}(0)\bar{S}_{22} + M_{12}(0)\bar{S}_{23}) + N_{22}(0)M_{13}(0)\bar{S}_{23} = 0\end{aligned} \tag{2.19}$$

$$\begin{aligned}&A_{c21}N_{12}(0) + A_{c22}N_{13}(0) + N'_{12}(0)A'_{c21} + N_{13}(0)A'_{c22} - T_3N_{22}(0) - N'_{22}(0)T'_3 \\ &\quad - T_4N_{23}(0) - N_{23}(0)T'_4 = 0 .\end{aligned} \tag{2.20}$$

From (2.19) we obtain

$$N_{12}(0) = N_{21}(0)W' + N_{11}(0)V'$$

where

$$W = A_{c22}^{-1}(T_3 + T_4V)$$

$$T_3 = \bar{S}_{13}M'_{22}(0) + \bar{S}'_{12}M_{21}(0) - \bar{S}_{23}M'_{12}(0) - \bar{S}'_{22}M_{11}(0)$$

$$T_4 = \bar{S}_{13}M_{23}(0) - \bar{S}_{23}M_{13}(0).$$

Substituting for $N_{12}(0)$, $M_{22}(0)$, $N_{22}(0)$ in (18), we get an equation identical to (A.1) where $N_{11}(0)$, $N_{21}(0)$, $M_{21}(0)$, $M_{11}(0)$, $F_{21}(0)$, $F_{22}(0)$, $M_{13}(0)$, $M_{12}(0)$ replace P_1, P_2, K_2, K_{11}, L_{21}, L_{22}, K_{13}, K_{12}, respectively.

Substituting for $N_{12}(0)$, $N_{23}(0)$, $M_{22}(0)$ in (2.20) and using equation (2.14), we get an equation identical to (A.5), where $N_{13}(0)$, $N_{11}(0)$, $N_{21}(0)$, $M_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace P_3, P_1, P_2, K_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22} respectively.

Decomposing (2.1e) and letting $\mu \to 0$, we have

$$R_{12}(F_{21}(0)N_{11}(0) + F_{22}(0)N'_{12}(0)) + R_{22}(F_{21}(0)N_{21}(0) + F_{22}(0)N'_{22}(0)) - B'_{12}M_{11}(0)N_{11}(0)$$
$$-B'_{22}(M'_{12}(0)N_{11}(0) + M_{13}(0)N'_{12}(0)) - B'_{12}M_{21}(0)N_{21}(0) - B'_{22}(M'_{22}(0)N_{21}(0) +$$
$$M_{23}(0)N'_{22}(0)) = 0 \tag{2.21}$$

$$(R_{12}F_{21}(0) - B'_{12}M_{11}(0) - B'_{22}M'_{12}(0))N_{12}(0) + (R_{22}F_{21}(0) - B'_{12}M_{21}(0) - B'_{22}M'_{22}(0))N_{22}(0)$$
$$+ (R_{12}F_{22} - B'_{22}M_{13})N_{13}(0) + (R_{22}F_{22}(0) - B'_{22}M_{23}(0)]N_{23}(0) = 0. \tag{2.22}$$

Substituting for $N_{12}(0)$, $N_{22}(0)$, $M_{22}(0)$, and $N_{23}(0)$ in equations (2.21) and (2.22) and using equation (2.14), we get equation (2.21) identical to (A.3) and equation (2.22) identical to (A.4) where $N_{11}(0)$, $N_{21}(0)$, $M_{21}(0)$, $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $F_{21}(0)$ $F_{22}(0)$, and $N_{13}(0)$ replace P_1, P_2, K_2, K_{11}, K_{12}, K_{13}, L_{21}, L_{22}, and P_3 respectively.

To compare the performance indices resulting from solving the full order problem with the ones resulting from the reduced order solution we need the following assumptions:

a. The fast optimization problem of the follower has a unique stabilizing solution. In other words there exists a unique K_{13} which is a solution of equation (2.3) for each L_{22} applied such that $\lambda(\hat{A}_{22}) < 0$.

b. The slow optimization problem of the follower has a unique solution after substituting for L_{12} from the fast problem, i.e. equations (2.6) and (2.7) have a unique solution for K_{11}, K_{12}.

c. The leader optimization problem has a unique stabilizing solution, i.e. there exists a unique pair L_{21} and L_{22} as a solution of the set of equations (2.3), (2.6), (2.7), (2.8) and (A.1) through (A.5) such that $\lambda(A_0) < 0$.

<u>Theorem</u>: If assumptions (a), (b), (c) are satisfied then:

(1) $\lim_{\mu \to 0} (u_i - u_{iap}) = 0 \qquad$ for $i = 1,2$

(2) $\lim_{\mu \to 0} (J_i - J_i^*) = 0$

where

$u_{iap} = -L_{i1}x - L_{i2}z, \qquad u_i = -F_{i1}x - F_{i2}z.$

J_i^* is the performance index when u_1 and u_2 are used.

J_i is the performance index when u_{1ap} and u_{2ap} are used.

Proof: (1) It was shown that $M_{11}(0)$, $M_{12}(0)$, $M_{13}(0)$, $M_{21}(0)$, $N_{11}(0)$, $N_{21}(0)$, $N_{13}(0)$, $F_{21}(0)$, $F_{22}(0)$ replace K_{11}, K_{12}, K_{13}, K_2, P_1, P_2, P_3, L_{21}, L_{22} respectively in the equations, and if the uniqueness assumptions are satisfied, then we have unique values of K_{11}, K_{12}, K_{13}, K_2, P_1, P_2, P_3, L_{21}, L_{22} and

$$K_{11} = M_{11}(0),\ K_{12} = M_{12}(0),\ K_{13} = M_{13}(0),\ F_{21}(0) = L_{21},\ F_{22}(0) = L_{22},\ K_2 = M_{21}(0),$$

$$P_1 = N_{11}(0),\ P_2 = N_{21}(0),\ P_3 = N_{13}(0).$$

For the follower:

$$u_{1ap} = -L_{11}x - L_{12}z .$$

Substituting for L_{11}, L_{12}, we obtain

$$u_{1ap} = -R_{11}^{-1}(B_{11}'K_{11} + B_{21}'K_{12}')x - R_{11}^{-1}B_{21}'K_{13}z .$$

But the exact control of the follower is:

$$u_1 = -R_{11}^{-1}\left[B_{11}' \quad \frac{B_{21}'}{\mu}\right] \begin{bmatrix} M_{11} & \mu M_{12} \\ \mu M_{12}' & \mu M_{13} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

$$= -R_{11}^{-1}[B_{11}'(M_{11}x + \mu M_{12}z) + B_{21}'(M_{12}'x + M_{13}z)]$$

$$= -R_{11}^{-1}[(B_{11}'M_{11} + B_{21}'M_{12}')x + \mu B_{11}'M_{12}z + B_{21}'M_{13}z] .$$

Clearly,

$$\lim_{\mu\to 0} u_1 = \lim_{\mu \to 0} u_{1ap}.$$

For the leader:

$$u_2 = -F_{21}x - F_{22}z$$

$$u_{2ap} = -L_{21}x - L_{22}z .$$

Clearly, $\lim_{\mu\to 0} u_2 = \lim_{\mu\to 0} u_{2ap}$.

(2) When the exact controls, $u_1 = -F_1 y$ and $u_2 = -F_2 y$ are used the resulted performance index $J_i^* = \frac{1}{2} y_o' M_i y_o$ where M_i is given by equations (2.1a), (2.1b). If u_{1ap}, and u_{2ap} are used, where

$$u_{1ap} = -R_{11}^{-1} B_1' K_1 y \;, \quad K_1 = \begin{bmatrix} K_{11} & 0 \\ \mu K_{12}' & \mu K_{13} \end{bmatrix}$$

$$u_{2ap} = -L_2 y$$

we will have J_i as the performance index, where $J_i = \frac{1}{2} y_o' W_i y_o$, and W_1, W_2 satisfy the following equations

$$W_1(A - S_{11}K_1 - B_2L_2) + (A - S_{11}K_1 - B_2L_2)'W_1 + Q_1 + K_1S_{11}K_1 + L_2'R_{12}L_2 = 0 \tag{2.23}$$

$$W_2(A - S_{11}K_1 - B_2L_2) + (A - S_{11}K_1 - B_2L_2)'W_2 + Q_2 + K_1S_{21}K_1 + L_2'R_{22}L_2 = 0 . \tag{2.24}$$

Subtracting (2.23) from (2.24) and (2.1a) from (2.1b) we find that

$P_1 = W_1 - M_1$ and $P_2 = W_2 - M_2$ satisfy

$$P_1(A - S_{11}K_1 - B_2L_2) + (A - S_{11}K_1 - B_2L_2)'P_1 + (K_1 - M_1)'S_{11}(K - M_1) + M_1B_2(L_2 - F_2)$$
$$+ (L_2 - F_2)'B_2'M_1 + L_2'R_{12}L_2 - F_2'R_{12}F_2 = 0 \tag{2.25}$$

$$P_2(A - S_{11}K_1 - B_2L_2) + (A - S_{11}K_1 - B_2L_2)\ P_2 + K_1S_{21}K_1 - M_1S_{21}M_1 - M_2S_{11}(K_1 - M_1)$$
$$-(K_1 - M_1)'S_{11}M_2 - M_2B_2(L_2 - F_2) - (L_2 - F_2)'B_2'M_2 + L_2'R_{22}L_2 - F_2'R_{22}F_2 = 0. \tag{2.26}$$

Taking $P_i = \begin{bmatrix} P_{i1} & \mu P_{i2} \\ \mu P_{i2}' & \mu P_{i3} \end{bmatrix}$

and substituting for P_1 and P_2 in equations (2.25) and (2.26) respectively, and setting $\mu \to 0$, we obtain

$$P_{i1}\hat{A}_{11} + P_{12}\hat{A}_{21} + \hat{A}_{11}'\hat{P}_{i1} + \hat{A}_{21}'P_{i2}' = 0 \tag{2.27}$$

$$P_{i1}\hat{A}_{12} + P_{i2}\hat{A}_{22} + \hat{A}_{21}'P_{i3} = 0 \tag{2.28}$$

$$P_{i3}\hat{A}_{22} + \hat{A}_{22}'P_{i3} = 0. \tag{2.29}$$

Since $\hat{A}_{22}$ is stable, $P_{i3} = 0$ is the unique solution of equation (2.29)

From (2.28) we have

$$P_{i2} = -[P_{i1}\hat{A}_{12}]\hat{A}_{22}^{-1}.$$

Substituting P_{i2} in (2.27), we obtain

$$P_{i1}(\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21}) + (\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})'P_{i1} = 0$$

or $P_{i1}A_0 + A_0P_{i1} = 0$. Since A_0 is stable, $P_{i1} = 0$. Thus

$$P_{ij} = 0 \quad \text{for } i = 1,2, \quad j = 1,2,3$$

<u>Remark 1</u>: In the LF game, the leader announces his strategy first, and the follower reacts by playing optimally, i.e., the follower chooses a strategy which lies on his reaction curve. So if the leader uses u_{2ap}, then the follower has to respond by choosing $u_1^0 = u_1\ (u_{2ap})$. In our case the follower does not choose u_1^0, but he chooses u_{1ap}, hence the rules of the game are violated. This deviation from the basic definition of the equilibrium solution of the game can be tolerated for computing the optimal strategies, if the resulting deviation of the leader's and follower's performance indices from their corresponding exact LF performance indices is small. In other words, the use of u_{1ap} by the

follower instead of u_1^0 can be tolerated as long as $J_1(u_{1ap}, u_{2ap})$ tends to $J_1(u_1(u_{2ap}), u_{2ap})$ as μ tends to zero. This can be easily shown to be the case by using a method which is equivalent to the one given in the theorem. We will not pursue this, any further in this chapter.

Remark 2: In this chapter and in [56], we consider a different information structure from the one given in [11]. In [11] the space of admissable strategies is restricted to be taken from the slow variables only, and the usual singular perturbation technique is used to obtain a well-posed solution. But in [56] and this chapter a more general information structure is considered, since both the slow and fast variables are available for measurement, and a new procedure, which depends on preservation of the information structure of the full order game, is used to obtain a well-posed solution.

CHAPTER 3

LEADER-FOLLOWER TEAM STRATEGIES FOR SINGULARLY PERTURBED SYSTEMS

3.1. Introduction

In this chapter, we develop a well-posed procedure to obtain low order and near optimal LF team equilibrium strategies for systems with slow and fast modes. In Section 3.2, we find the limiting behavior of the full order game as μ tends to zero. In Section 3.3, we solve two reduced order games. First, we give sufficient conditions for the existence of solutions for the fast LF team problem. Secondly, we give sufficient conditions for the existence of solutions for a hybrid slow (hybrid because the fast gains are imbedded) LF team problem. The basic property of this reduced order hybrid slow game is that it has the same information structure as that of the full order one. In Section 3.4, we show that, under certain conditions, the reduced order games and the full order one are equivalent in the limit as μ tends to zero. In Section 3.5, we apply our well-posed procedure to the case when the leader uses strategies with finite dimensional memory and solve a numerical example.

3.2. Full Order Problem

Let us consider the singularly perturbed linear time-varying system

$$\dot{x} = A_{11}(t)x + A_{12}(t)x + B_{11}(t)u_1 + B_{12}(t)u_2; \qquad x(0) = x_o$$

$$\mu\dot{z} = A_{21}(t)x + A_{22}(t)z + B_{21}(t)u_1 + B_{22}(t)u_2; \qquad z(0) = z_o$$

where $x \in R^{n_1}$; $z \in R^{n_2}$, $u_i \in R^{m_i}$ for $i = 1,2$, μ is a small positive parameter, and A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22} are continuous in t for all $t \in [t_o, t_f]$. Assume that the ith player wishes to minimize the following function

$$J_i(u_i,u_j) = \frac{1}{2}\int_{t_o}^{t_f} (y'Q_i(t)y + u_i'R_{ii}(t)u_i + u_j'R_{ij}u_j)dt; \quad \text{for } i=1,2 \quad i\neq j$$

where $y = \begin{bmatrix} x \\ z \end{bmatrix}$; $Q_i(t) = \begin{bmatrix} Q_{i1}(t) & Q_{i2}(t) \\ Q_{i2}'(t) & Q_{i3}(t) \end{bmatrix}$, which is a symmetric positive semi-definite matrix, and R_{ii}, R_{ij} are symmetric positive definite matrices. Let us take player one as the leader and player two as the follower. The procedure to solve the team leader-follower game, with memory in the control structure is as follows [18]:

1. Solve the leader's problem as a control problem with $J_1(u_1,u_2)$ as the objective function, and u_1,u_2 as controls under the constraint of the state equation. The optimal controls are $(\bar{u}_1,\bar{u}_2)$, where

$$\bar{u}_1(t) = -R_{11}^{-1}B_1'Ky$$
$$\bar{u}_2(t) = -R_{12}^{-1}B_2'Ky$$

and K satisfies the following Riccati equation

$$\dot{K} + KA + A'K + Q_1 - K[B_1R_{11}^{-1}B_1' + B_2R_{12}^{-1}B_2']K = 0$$

and

$$K(t_f) = 0.$$

Define $\bar{y}$ (t) as the resulting trajectory when the controls $\bar{u}_1(t), \bar{u}_2(t)$ are applied.

2. Consider the function $\tilde{u}_1$, which is represented by a Lebesgue-Stieltjes integral of the following form

$$\tilde{u}_1(t) = \int_{t_o}^{t} d_s\eta(t,s)y(s),$$

such that

$$\tilde{u}_1(t) = \bar{u}_1(t).$$

Then with $\tilde{u}_1$ in the objective function $(J_2(\tilde{u}_1,u_2))$, of the follower and in the state equation, minimize $J_2(\tilde{u}_1,u_2)$ with respect to u_2 and find conditions such that if the optimal $(\tilde{u}_1,\bar{u}_2)$ are applied then the resulting trajectory will be $\bar{y}\ (t)$. If these conditions are satisfied then $(\tilde{u}_1,\bar{u}_2)$ constitute a leader-follower strategy pair. These conditions are stated in [18], but we restate them here for completeness.

If there exists a function $\eta(t,\theta)$ with $\eta(t,\theta)=0$ for $\theta \geq t$ and $(n_1+n_2)\times(n_1+n_2)$ matrix P which satisfy

$$\int_{t_o}^{t} d_s\eta(t,s)y\ (s) = -R_{11}^{-1}B_1'K(t)y\ (t) \qquad t\in[t_o,t_f] \tag{3.2}$$

$$R_{22}^{-1}(t)B_2'(t)P(t) = R_{12}^{-1}(t)B_2'(t)K(t) \qquad t\in[t_o,t_f] \tag{3.3}$$

$$P(t)\bar{y}\ (t) - \int_{t}^{t_f} (Q_2 + A'P + \eta'(\tau,t)F(\tau))\bar{y}\ (\tau)d\tau = 0 \qquad t\in[t_o,t_f] \tag{3.4}$$

where

$$F(t) = R_{21}(t)R_{11}^{-1}(t)B_1'(t)K(t)-B_1'(t)P(t) \tag{3.5}$$

and $\bar{y}\ (t)$ satisfies the following linear differential equation

$$\dot{\bar{y}} = (A(t)-B_1R_{11}^{-1}B_1'K-B_2R_{22}^{-1}B_2'P)\bar{y}\ (t) \tag{3.6}$$

$$\bar{y}(t_o) = y_o$$

then

$$\tilde{u}_1(t) = \int_{t_o}^{t} d_s\eta(t,s)y(s)^{\dagger}$$

$$u_2(t) = -R_{12}^{-1}B_2'(t)K(t)y(t)$$

are optimal leader-follower strategies. Furthermore they are also the team solution.

Preliminary analysis indicates that the following forms are useful for our problem

$$K(t,\mu) = \begin{bmatrix} K_1(t,\mu) & \mu K_2(t,\mu) \\ \mu K_2'(t,\mu) & \mu K_3(t,\mu) \end{bmatrix} \quad ; \quad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}' & Q_{i3} \end{bmatrix}$$

$$\eta(t,s,\mu) = [\eta_1(t,s,\mu) \quad \eta_2(t,s,\mu)]$$

$$F(t,\mu) = [F_1(t,\mu) \quad F_2(t,\mu)]$$

$$P(t,\mu) = \begin{bmatrix} P_1(t,\mu) & \mu P_2(t,\mu) \\ \mu P_3(t,\mu) & \mu P_4(t,\mu) \end{bmatrix}.$$

We substitute the forms of $K(t,\mu)$, Q_i, $\eta(t,s,\mu)$, $F(t,\mu)$, and $P(t,\mu)$ as given above in equations (3.1)-(3.6), decompose them, and take the limit as μ tends to zero. The reader is referred to Appendix B for the resulting equations.

Several remarks are made regarding the above procedure.

R1: The state vector $y = \begin{bmatrix} x \\ z \end{bmatrix}$ in equations (3.2), (3.4), and (3.6) is decoupled into fast and slow subvectors by using a transformation due to Chang [25].

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†For a precise description of the function $\eta(t,s)$ the reader is referred to [18].

So the control memory of the leader is decomposed into a memory of the slow modes and a memory of the fast modes.

R2: In general the matrix $P(t,\mu)$ is nonsymmetric, but it is symmetric for some special $\eta(t,s)$'s.

R3: After decomposing equation (3.4), and letting $\mu \to 0^+$, we get as one of the equations

$$\int_t^{t_f} (Q_{22} + A_{21}' P_4(\tau,0) + \eta_1'(\tau,t,0) F_2(\tau,0)) \Psi_{22}(\tau,t,0) d\tau = 0.$$

But since the state transition matrix has full rank the above equation implies (B.16), and the same situation applies in (B.17) and (B.18).

3.3. Reduced Order Games

The class of leader-follower games, which is considered in the previous section, will also be considered in this section but for the case in which the reduced order systems are given. The fast subsystem is

$$\mu \dot{z}_f = A_{22} z_f + B_{21} u_{1f} + B_{22} u_{2f}$$

and the fast part of the objective function is

$$J_{if} = \frac{1}{2} \int_{t_o}^{t_f} (z_f' Q_{i3} z_f + u_{if}' R_{ii} u_{if} + u_{jf}' R_{ij} u_{jf}) dt.$$

Sufficient conditions for the existence of an optimal leader-follower fast strategy pair with memory which coincides with the team fast strategy pair are given in Appendix C.

If we let $\mu \to 0^+$ in the original system, then

$$\dot{x}_s = A_{11} x_s + A_{12} z_s + B_{11} u_{1s} + B_{12} u_{2s}$$

$$0 = A_{21} x_s + A_{22} z_s + B_{21} u_{1s} + B_{22} u_{2s}.$$

Assuming that $A_{22}(t)$ is nonsingular for all $t\in[t_o,t_f]$, then

$$z_s = -A_{22}^{-1}(A_{21}x_s + B_{21}u_{1s} + B_{22}u_{2s})$$

Eliminating z_s, we have

$$\dot{x}_s = \hat{A}_{11}x_s + \hat{B}_{11}u_{1s} + \hat{B}_{12}u_{2s}$$

where

$$\hat{A}_{11} = A_{11}-A_{12}A_{22}^{-1}A_{21}; \quad \hat{B}_{11} = B_{11}-A_{12}A_{22}^{-1}B_{21}$$

$$\hat{B}_{12} = B_{12}-A_{12}A_{22}^{-1}B_{22}.$$

The slow part of the objective function is

$$J_{is} = \frac{1}{2}\int_{t_o}^{t_f}[x_s'\tilde{Q}_{i1}x_s + x_s'\tilde{Q}_{i2}u_{is} + u_{is}'\tilde{Q}_{i2}'x_s + x_s'\bar{Q}_{i2}u_{js} + u_{js}'\bar{Q}_{i2}'x_s + u_{is}'\tilde{R}_{ii}u_{is}$$
$$+ u_{js}'\tilde{R}_{ij}u_{js} + u_{is}'\bar{Q}_{i3}u_{js} + u_{js}'\bar{Q}_{i3}'u_{is}]dt$$

where

$$\tilde{Q}_{i1} = Q_{i1}-Q_{i2}A_{22}^{-1}A_{21}-(A_{22}^{-1}A_{21})'Q_{i2}' + (A_{22}^{-1}A_{21})'Q_{i3}A_{22}^{-1}A_{21}$$

$$\tilde{Q}_{i2} = -Q_{i2}A_{22}^{-1}B_{2i} + (A_{22}^{-1}A_{21})'Q_{i3}A_{22}^{-1}B_{2i}$$

$$\bar{Q}_{i2} = -Q_{i2}A_{22}^{-1}B_{2j} + (A_{22}^{-1}A_{21})'Q_{i3}A_{22}^{-1}B_{2j}$$

$$\bar{Q}_{i3} = B_{2i}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2j}$$

$$\tilde{R}_{ij} = R_{ij} + B_{2j}'(A_{22}^{-1})'Q_{i3}A_{22}^{-1}B_{2j}.$$

Sufficient conditions for the existence of the leader-follower team hybrid slow solution are obtained as in [18]. We have the following facts and conditions.

1. $(\bar{u}_{1s},\bar{u}_{2s})$ are the team slow optimal controls, where

$$\bar{u}_{1s} = -\tilde{R}_{11}^{-1}[\tilde{Q}'_{12}x_s + \hat{B}'_{11}K_s x_s + \tilde{Q}_{13}\bar{u}_{2s}]$$

$$\bar{u}_{2s} = -\tilde{R}_{12}^{-1}[\bar{Q}'_{12}x_s + \bar{Q}'_{13}\bar{u}_{1s} + \hat{B}'_{12}K_s x_s]$$

where $\bar{x}_s$ is the resulting trajectory when u_{1s}, u_{2s} are applied.

$\bar{u}_{1s}$ and $\bar{u}_{2s}$ can be expressed in the following form

$$\bar{u}_{is} = -M_{is}x_s, \qquad \text{for } i=1,2,$$

and K_s satisfies

$$\dot{K}_s + \tilde{Q}_{11} + K_s(\hat{A}_{11} - \hat{B}_{11}M_{1s} - B_{12}M_{2s}) + (\hat{A}_{11} - \hat{B}_{11}M_{1s} - \hat{B}_{12}M_{2s})'K_s - \tilde{Q}_{12}M_{1s} - M'_{1s}\tilde{Q}'_{12}$$

$$-\bar{Q}_{12}M_{2s} - M'_{2s}\bar{Q}'_{12} + M'_{1s}\tilde{R}'_{11}M_{1s} + M'_{2s}\tilde{R}_{12}M_{2s} + M'_{1s}\bar{Q}_{13}M_{2s} + M'_{2s}\bar{Q}'_{1s}M_{1s} = 0$$

$$K_s(t_f) = 0.$$

2. The leader applies a control $\tilde{u}_{1s}$ of the following form

$$\tilde{u}_{1s}(t) = \int_{t_o}^{t}(d_s\eta_s(t,s) - d_s\eta_f(t,s,0)A_{22}^{-1}A_{21})x_s(s) - \int_{t_o}^{t} d_s\eta_f(t,s,0)A_{22}^{-1}B_{21}u_{1s}(s)$$

$$-\int_{t_o}^{t} d_s\eta_f(t,s,0)A_{22}^{-1}B_{22}u_{2s}(s).$$

This specific form of the leader's control is chosen so as to preserve the information structure of the full order problem in the limit as μ tends to zero, since a closer look at its form reveals that $\tilde{u}_{1s}$ can be expressed as

$$\tilde{u}_{1s} = \int_{t_o}^{t} d_s\eta_s(t,s)x_s(s) + \int_{t_o}^{t} d_s\eta_f(t,s)z_s(s)$$

where z_s is the slow part of the z-state vector. In the full order problem the leader has access to the past history of the trajectory, but in the reduced order problem, he has access to the history of the slow part of

the trajectory, his strategies and the strategies of the follower. Also, the leader is not using a standard slow control since he is using the fast gain $\tilde{\eta}_f(t,s)$ which is obtained from the solution of the fast game. The use of the fast gain is necessitated by the preservation of the information structure.

3. Sufficient conditions for the existence of the hybrid slow leader-follower solution are given in Appendix D.
4. If η_f is not used in 2 above, the equations of the reduced order games will not correspond to the full order ones, as will be clearly explained, later in this chapter.

3.4. The Correspondence Between the Reduced Order and the Full Order Games

This section contains four lemmas which show the correspondence between the reduced and the full order games, and prepare for the main theorem which describes the procedure to obtain a well-posed and near optimal strategies depending on the reduced order subsystems.

Before stating the lemmas, let us form the composite controls $u_{1c}(t)$, $u_{2c}(t)$ as

$$u_{ic}(t) = u_{is} + u_{if} + 0(\mu), \quad \text{for } i=1,2$$

or

$$u_{1c}(t) = -M_{1s}x_s - R_{11}^{-1}B_{21}'K_{1f}(t,0)z_f(t) + 0(\mu)$$

$$u_{2c}(t) = -M_{2s}x_s - R_{12}^{-1}B_{22}'K_{1f}(t,0)z_f(t) + 0(\mu).$$

After using the same manipulations as in [9], we get u_{1c}, u_{2c} of the following form

$$u_{1c} = -R_{11}^{-1}[B_{11}' \quad \frac{B_{21}'}{\mu}] \begin{bmatrix} K_s(t) & 0 \\ \mu K_m'(t) & \mu K_{1f}(t,0) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

$$u_{2c} = -R_{12}^{-1}[B_{12}' \quad \frac{B_{22}'}{\mu}] \begin{bmatrix} K_s(t) & 0 \\ \mu K_m'(t) & \mu K_{1f}(t,0) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

where

$$K_m(t) = -\hat{A}_{22}^{-1}[K_sA_{12} + A_{21}'K_{1f}(t,0) + Q_{12} - K_s\bar{S}_2K_{1f}(t,0)]$$

$$\hat{A}_{22} = A_{22} - \bar{S}_3K_{1f}(t,0)$$

$$\bar{S}_2 = B_{11}R_{11}^{-1}B_{21}' + B_{12}R_{12}^{-1}B_{22}'$$

$$\bar{S}_3 = B_{21}R_{11}^{-1}B_{21}' + B_{22}R_{12}^{-1}B_{22}'.$$

In [17], it was shown that if

1. $A_{11},A_{12},A_{21},A_{22},B_{11},B_{12},B_{21},B_{22}$ are continuous in t,
2. R_{ii},R_{ij},Q_i are continuous in t for i=1,2, j=1,2, and i≠j,
3. Equation (C.4) has a unique solution as ~ tends to zero.
4. The fast subsystem is asymptotically stable,

then

$$\lim_{\mu\to 0}[K_3(t,\mu)-K_{1f}(t,\mu)] = 0 \qquad \forall t\in[t_o,t_f]$$

$$\lim_{\mu\to 0}[K_2(t,\mu)-K_m(t)] = 0 \qquad \forall t\in[t_o,t_f)$$

$$\lim_{\mu\to 0}[K_1(t,\mu)-K_s(t)] = 0 \qquad \forall t\in[t_o,t_f]$$

and as a result we have

$$\lim_{\mu \to 0}[u_i(t,\mu)-u_{ic}(t,\mu)] = 0 \quad \text{for } i=1,2, \text{ and for all } t\in[t_o,t_f)$$

where

$$u_1(t,\mu) = -R_{11}^{-1}B_1'Ky$$

$$u_2(t,\mu) = -R_{12}^{-1}B_2'Ky.$$

The relevance of these results to our problem will be clear later.

<u>Lemma 3.1:</u> Let

$$G_o = \begin{bmatrix} -M_{1s}(t) \\ -M_{2s}(t) \end{bmatrix} ; \qquad \begin{aligned} \bar{B}_1 &= [B_{11} \quad B_{12}] \\ \bar{B}_2 &= [B_{21} \quad B_{22}] \end{aligned}$$

$$G_2 = \begin{bmatrix} -R_{11}^{-1}B_{21}'K_{1f}(t,0) \\ -R_{12}^{-1}B_{22}'K_{1f}(t,0) \end{bmatrix} ; \qquad u_c = \begin{bmatrix} u_{1c} \\ u_{2c} \end{bmatrix}$$

$$u_s = G_o x_s ; \qquad u_f = G_2 z_f .$$

Form

$$u_c = [(I + G_2A_{22}^{-1}\bar{B}_2)G_o + G_2A_{22}^{-1}A_{21}]x + G_2z.$$

If u_s, u_f, u_c are applied to the slow, fast, and the full order systems respectively, and if $(A_{22}+\bar{B}_2G_2)$ is asymptotically stable, then

$$\lim_{\mu \to 0}(x(t)-x_s(t)) = 0 \qquad \forall t\in[t_o,t_f]$$

$$\lim_{\mu \to 0}(z(t) + A_{22}^{-1}(A_{21}+\bar{B}_2G_o)x_s(t)-z_f(t)) = 0 \qquad \forall t\in[t_o,t_f)$$

where $x_s(t), z_f(t)$ are the resulting trajectories after applying u_s and u_f to the slow and the fast subsystems respectively.

<u>Proof</u>: When we apply u_c to the full order system we get

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11}+\bar{B}_1(I+G_2A_{22}^{-1}\bar{B}_2)G_o+\bar{B}_1G_2A_{22}^{-1}A_{21} & A_{12}+\bar{B}_1G_2 \\ \dfrac{(A_{22}+\bar{B}_2G_2)A_{22}^{-1}(A_{21}+\bar{B}_2G_o)}{\mu} & \dfrac{A_{22}+\bar{B}_2G_2}{\mu} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.$$

It is clear that the present system is exactly equivalent to the system described by equation (B.8) before, since equal controls are applied to the full order system. Using the usual singular perturbation techniques, see [17], we get

$$\lim_{\mu\to 0}(x(t)-x_s(t)) = 0 \qquad \forall t\in[t_o,t_f]$$

$$\lim_{\mu\to 0}(z(t)+A_{22}^{-1}(A_{21}+\bar{B}_2G_o)x_s(t)-z_f(t)) = 0 \qquad \forall t\in[t_o,t_f). \qquad \blacksquare$$

After noting the equivalence between the present system and the system described by equation (B.8), it is easy to see that

$$\lim_{\mu\to 0}(\psi_{11}(s,t,\mu)-\phi_s(s,t)) = 0 \qquad \forall t\in[t_o,t_f]$$

$$\lim_{\mu\to 0}(\psi_{22}(s,t,\mu)-\phi_f(s,t,\mu)) = 0 \qquad \forall t\in[t_o,t_f]$$

where $\psi_{11}(s,t,\mu)$ and $\psi_{22}(s,t,\mu)$ are the state transition matrices described by equations (B.11) and (B.12) respectively.

Lemma 3.2: If in the limit as μ tends to zero, equation (C.4) has a unique solution, and there exist unique values for $\eta_f(t,s,0)$, $K_{2_f}(t,0)$ which satisfy equations (C.1), (C.2), and (C.3), and if the fast subsystem is asymptotically stable i.e. $\lambda(\tilde{A}_{22}) < 0$ then

$$K_{1_f}(t,0) = K_3(t,0); \qquad \eta_f(t,s,0) = \eta_2(t,s,0); \qquad K_{2_f}(t,0) = P_4(t,0)$$

$$F_f(t,0) = F_2(t,0); \qquad \phi_f(\tau,t,0) = \psi_{22}(\tau,t,0)$$

where K_3, η_2, P_4, F_2, and ψ_{22} are defined in Section 3.2 and Appendix B.

Proof: If we let $\mu \to 0^+$ in equations (C.1), (C.2), (C.3), (C.4), (C.5), and (C.6) of the fast game, and then compare them with (B.14), (B.5), (B.18), (B.3), (B.7), and (B.12) of the full-order game, we notice that the first set of equations are exactly equivalent to the second set, where $K_{1_f}(t,0)$, $\eta_f(t,s,0)$, $K_{2_f}(t,0)$, $F_f(t,0)$, $\phi_f(\tau,t,0)$ replace $K_3(t,0)$, $\eta_2(t,s,0)$, $P_4(t,0)$, $F_2(t,0)$, $\psi_{22}(\tau,t,0)$ respectively. The uniqueness assumptions stated in the lemma are sufficient for equality of the above terms. The stability condition is necessary and sufficient for the boundedness of $\phi_f(\tau,t,0)$.

Lemma 3.3: If the assumptions of Lemma 3.2 are satisfied, and if there exists unique values of $\eta_s(t,s)$, $P_s(t)$ which satisfy equations (D.1), (D.2), (D.3) and

$$Q_{22} + A_{21}'K_{2_f}(\tau,0) + \eta_s'(\tau,t)F_f(\tau,0) = 0 \;\ldots \qquad \forall t \leq \tau \tag{3.12}$$

then

$$P_1(t,0) = P_s(t); \quad \eta_1(\tau,t,0) = \eta_s(\tau,t) \quad \text{and } d_t\eta_1(\tau,t,0) = d_t\eta_s(\tau,t)^{(*)}$$

where P_1 and η_1 are defined in Appendix B.

Proof: Comparing equation (B.6) of the full order game with equation (D.4) we notice that by adding the term $(F_f - R_{21}R_{11}^{-1}B_{21}'K_{1_f} - B_{21}'K_{2_f})A_{22}^{-1}(A_{21} - B_{21}M_{1s} - B_{22}M_{2s})$,

---

*In general if $f_n(t)$, $f(t)$ are differentiable almost everywhere (a.e.) for all n, and $f_n(t) \to f(t)$ pointwise, this does not imply that $\frac{df_n(t)}{dt} \to \frac{df(t)}{dt}$ a.e. for example, take $f_n(t) = \frac{1}{\sqrt{n}} \sin nt$.

which is equal to zero, to equation (D.4), we get an equation equivalent to (B.6), after using the identity

$$M_{1s} + R_{11}^{-1}B_{21}'K_{1_f}(t,0)A_{22}^{-1}(A_{21}-B_{21}M_{1s}-B_{22}M_{2s}) = R_{11}^{-1}(B_{11}'K_1(t,0) + B_{21}'K_2'(t,0)),$$

and substituting for $P_3(t,0) = -(A_{22}^{-1})'[Q_{22}'+A_{12}'P_1(t,0)+\eta_2'F_1(t,0)]$ obtained from equation (B.17) in (B.6). So $F_1(t,0)$, $P_1(t,0)$, $\eta_2(\tau,t,0)$, $F_2(t,0)$ in (B.6) replace $F_s+F_fA_{22}^{-1}(A_{21}-B_{21}M_{1s}-B_{22}M_{2s})P_s(t)$, $\eta_f(\tau,t,0)$, $F_f(t,0)$ respectively in (D.4).

Comparing equation (B.4) with equation (D.2), we see that by using the identity

$$R_{12}^{-1}(B_{12}'K_1(t,0)+B_{22}'K_2'(t,0)) = M_{2s} + R_{22}^{-1}B_{22}'K_{2_f}(t,0)A_{22}^{-1}(A_{21}-B_{21}M_{1s}-B_{22}M_{2s})$$

and by substituting for the value of $P_3(t,0)$ from (B.17) in equation (B.4), the two equations will be equivalent, where $P_1(t,0)$, $F_1(t,0)$, $\eta_2(\tau,t,0)$, $F_2(t,0)$, $P_4(t,0)$ replace $P_s(t)$, $F_s+F_fA_{22}^{-1}(A_{21}-B_{21}M_{1s}-B_{22}M_{2s})$, $\eta_f(\tau,t,0)$, $F_f(t,0)$, and $K_{2_f}(t,0)$ respectively.

Equations (D.1) and (B.13) will be equivalent after using some of the previous identities, and we will have $d_s\eta_1(t,s,0)$ and $d_s\eta_2(t,s,0)$ replace $d_s\eta_s(t,s)$ and $d_s\eta_f(t,s,0)$ respectively.

If we substitute for $P_3(t,0)$ using (B.17), and $P_4(t,0)$ using (B.18) in equation (B.15), we notice that equation (B.15) is equivalent to equation (D.3) where $P_1(t,0)$, $F_1(t,0)$, $\bar{\eta}_1(s,t,0)$, $\bar{\eta}_2(s,t,0)$ replace $P_s(t)$, $F_s+F_fA_{22}^{-1}(A_{21}-B_{21}M_{1s}-B_{22}M_{2s})$, $\eta_s(s,t)$ and $\eta_f(s,t,0)$ respectively.

If $\eta_s'(\tau,t)$ also satisfies equation (3.12), which is equivalent to equation (B.16), then under the uniqueness assumption given in the lemma, we have

$$\eta_1(s,t,0) = \eta_s(s,t)$$

$$P_1(t,0) = P_s(t)$$

$$d_s\eta_1(t,s,0) = d_s\eta_s(t,s).$$

Before stating the main theorem, we need the following definition and one more lemma.

Definition: A function $f(t,\mu)$ is said to be well behaved in μ, if there exists an integrable function $g(t)$ such that for all μ, $f(t,\mu) \leq g(t)$ for almost all t.

Let $\phi_o(t,t_o,\mu)$ be the state transition matrix which satisfies

$$\dot{\phi}_o(t,t_o,\mu) = A(t)\phi_o(t,t_o,\mu); \qquad \phi(t_o,t_o,\mu) = I$$

where

$$A(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ \dfrac{A_{21}(t)}{\mu} & \dfrac{A_{22}(t)}{\mu} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \bar{B}_1 \\ \dfrac{\bar{B}_2}{\mu} \end{bmatrix}.$$

Lemma 3.4: If $A_{22}(t)$ is asymptotically stable $\forall t \in [t_o, t_f]$, then $\lim_{\mu \to 0} \phi(t,t_o,\mu)B(\mu)$ is bounded for all $t > t_o$.

Proof: See Appendix E:

Theorem: If the assumptions of Lemmas 3.2 and 3.3 are satisfied, and $d_s\eta_1(t,s,\mu)$, $d_s\eta_2(t,s,\mu)$, $u_1(t,\mu)$, $u_2(t,\mu)$ are well behaved in μ, and if the leader and the follower choose u_{1ap}, u_{2ap} as their controls, where

$$u_{1ap}(t) = \int_{t_o}^{t} d_s\eta_s(t,s)x(s) + \int_{t_o}^{t} d_s\eta_f(t,s,0)z(s)$$

$$u_{2ap}(t) = -R_{12}^{-1}[B_{12}' \quad \frac{B_{22}'}{\mu}] \begin{bmatrix} K_s(t) & 0 \\ \mu K_m'(t) & \mu K_{1f}(t,0) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

then

1) $\lim_{\mu \to 0}(u_1(t,\mu) - u_{1ap}(t,\mu)) = 0 \qquad \forall t \in [t_o, t_f]$

$\lim_{\mu \to 0}(u_2(t,\mu) - u_{2ap}(t,\mu)) = 0 \qquad \forall t \in [t_o, t_f)$

2) $\lim_{\mu \to 0}(y(t,\mu) - y_{ap}(t,\mu)) = 0 \qquad \forall t \in [t_o, t_f]$

3) $\lim_{\mu \to 0}(J_i(\mu) - J_{iap}(\mu)) = 0 \qquad$ for $i=1,2$

where $y(t,\mu)$ and $J_i(\mu)$ for $i=1,2$ are the resulting trajectory and objective functions from applying $u_1(t,\mu)$, $u_2(t,\mu)$, while $y_{ap}(t,\mu)$ and $J_{iap}(\mu)$ for $i=1,2$ are the resulting trajectory and objective functions from applying $u_{1ap}(t,\mu)$, $u_{2ap}(t,\mu)$.

Proof:

1. For the follower (player 2), it was proved before that

$$\lim_{\mu \to 0}(u_2(t,\mu) - u_{2ap}(t,\mu)) = 0 \qquad \forall t \in [t_o, t_f).$$

For the leader

$$\lim_{\mu\to 0}(u_1(t,\mu)-u_{1ap}(t,\mu)) = \lim_{\mu\to 0}\int_{t_o}^{t}(d_s\eta_1(t,s,\mu)-d_s\eta_s(t,s))x(s)$$

$$+\lim_{\mu\to 0}\int_{t_o}^{t}(d_s\eta_2(t,s,\mu)-d_s\eta_f(t,s,0))z(s)$$

since

$$d_s\eta_1(t,s,\mu)\to d_s\eta_s(t,s), \quad \text{pointwise}$$

$$d_s\eta_2(t,s,\mu)\to d_s\eta_f(t,s,0), \text{ pointwise}$$

and since $\lim_{\mu\to 0} x(s)$ and $\lim_{\mu\to 0} z(s)$ are bounded, because of the stability assumption in Lemma 3.2, and since $d_s\eta_1(t,s,\mu)$, $d_s\eta_2(t,s,\mu)$ are well behaved in μ, then by the Lebesgue Convergence Theorem (LCT), we get

$$\lim_{\mu\to 0}(u_1(t,\mu)-u_{1ap}(t,\mu)) = 0 \qquad \forall t\in[t_o,t_f].$$

2. Let $y=\begin{bmatrix} x \\ z \end{bmatrix}$; $u=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$y(t,\mu) = \phi(t,t_o,\mu)y(0)+\int_{t_o}^{t}\phi(t,\tau,\mu)B(\mu)u(\tau,\mu)d\tau$$

$$y_{ap}(t,\mu) = \phi(t,t_o,\mu)y(0)+\int_{t_o}^{t}\phi(t,\tau,\mu)B(\mu)u_{ap}(\tau,\mu)d\tau$$

$$\lim_{\mu\to 0}(y(t,\mu)-y_{ap}(t,\mu)) = \lim_{\mu\to 0}\int_{t_o}^{t}\phi(\tau,t,\mu)B(\mu)(u(\tau,\mu)-u_{ap}(\tau,\mu))d\tau$$

since $\phi(\tau,t,\mu)B(\mu)$ is bounded $\forall\mu$ and

$$\lim_{\mu\to 0}(u(\tau,\mu)-u_{ap}(\tau,\mu)) = 0$$

then using LCT, we get

$$\lim_{\mu\to 0}(y(t,\mu)-y_{ap}(t,\mu)) = 0.$$

3. If $u_i(t,\mu)$ for $i=1,2$ are applied to the system, the resultiong objective function of the leader is

$$J_1(\mu) = \frac{1}{2}\int_{t_o}^{t_f} (y'Q_1 y + u_1'R_{11}u_1 + u_2'R_{12}u_2)dt.$$

If $u_{iap}(t,\mu)$ for $i=1,2$ are applied to the system, the resulting objective function of the leader is

$$J_{1ap}(\mu) = \frac{1}{2}\int_{t_o}^{t_f} (y_{ap}'Q_1 y_{ap} + u_{1ap}'R_{11}u_{1ap} + u_{2ap}'R_{12}u_{2ap})dt$$

$$\begin{aligned}(J_1(\mu)-J_{1ap}(\mu)) &= \frac{1}{2}\int_{t_o}^{t_f} y'Q_1(y-y_{ap})dt + \frac{1}{2}\int_{t_o}^{t_f} y_{ap}'Q_1(y-y_{ap})dt \\ &+ \frac{1}{2}\int_{t_o}^{t_f} u_{11}'R_{11}(u_1-u_{1ap})dt + \frac{1}{2}\int_{t_o}^{t_f} u_{1ap}'R_{11}(u_1-u_{1ap})dt \\ &+ \frac{1}{2}\int_{t_o}^{t_f} u_2'R_{12}(u_2-u_{2ap}) + \frac{1}{2}\int_{t_o}^{t_f} u_{2ap}'R_{12}(u_2-u_{2ap})dt.\end{aligned}$$

Using the results of (1) and (2), and the well-behavedness property, and then applying LCT we get

$$\lim_{\mu\to 0}(J_1(\mu)-J_{1ap}(\mu)) = 0.$$

The same procedure can be applied for the follower's objective function, and we get

$$\lim_{\mu\to 0}(J_2(\mu)-J_{2ap}(\mu)) = 0. \qquad \blacksquare$$

The theorem claims that under proper assumptions if the leader obtains $\eta_f(t,s)$ and $\eta_s(t,s)$ from the solution of the fast and hybrid slow games respectively, and applies u_{1ap} to the full order system, and if the follower uses u_{2ap}, as described above, then the resulting objective functions of both players are well-posed.

If the fast game is solved as before, and the slow game is solved independently of the fast information, i.e., the leader applies $\tilde{u}_{1s}$ $\tilde{u}_{1s} = \int_{t_0}^{t} d_s \bar{\eta}_s(t,s) x_s(s)$, then the sufficient conditions for the existence of the reduced order LF team strategies may be satisfied, while these conditions are not satisfied for the full order LF team strategies (as will be shown in the next section). This makes such a procedure undesirable. But if we can show that the sufficient conditions for existence of the full order and the reduced order LF team solutions are satisfied, then this procedure, in which the fast and the slow information are decoupled, is well-posed. This procedure is exactly equivalent to the one used in control problems. The basic feature of our procedure, which depends on the preservation of the information structure of the full order game, is that the conditions of the reduced order games are equivalent, under certain assumptions, to the full order ones, in the limit as the small parameter μ tends to zero. This feature makes our procedure more general and more desirable than the other one.

3.5. The Case of a LF Team Strategy with Finite-Dimensional Memory

In the previous sections we have assumed that the leader uses a strategy which is described by a Lebesgue-Stieltjes measure. This general strategy provides the leader with much flexibility in enforcing his team solution. The sufficient conditions under which the leader can enforce his team solution, by using such a strategy, are described by intgro-differential equations which are very difficult to solve, hence such forms of strategies are unappealing in applications. Fortunately, it is possible to greatly decrease the mathematical complication of the sufficient

conditions, by allowing the leader to adopt a slightly less flexible and less general strategy. In particular, he can choose the following representation of his team strategy:

$$\gamma_1(t,x(s),s \le t) = -R_{11}^{-1}B_1'Ky(t) + P'(y(t) - \bar{y}(t)) + T'(W(t) - \bar{W}(t)) \tag{3.13}$$

where W(t) is an n-dimensional vector function which satisfies

$$\dot{W} = \bar{A}W + \bar{C}y + \bar{B}k_1 + \bar{D}u_2 \qquad W(0) = 0$$

and $\bar{W}$ is the solution of the above equation with y, u_1, and u_2 are replaced by their optimal team forms. The leader has the freedom to choose the matrix functions $\bar{A}$, $\bar{C}$, $\bar{B}$, $\bar{D}$, P and T which gives him a high degree of flexibility.

If the leader announces the strategy (3.13), then the follower's reaction to that can easily be found, and the sufficient conditions for the team solution to exist can be obtained, by basically following the same procedure described in section (3.2). Sufficient conditions for existence of the LF team solution are given in the following proposition, which we state here without proof.

Proposition 3.1: If there exist appropriate dimensional matrix functions $\bar{A}(\cdot)$, $\bar{C}(\cdot)$, $\bar{B}(\cdot)$, $\bar{D}(\cdot)$, $P(\cdot)$ and $T(\cdot)$ so that the following holds:

$$R_{22}^{-1}(B_2'M + \bar{D}'N) = R_{12}^{-1}B_2'K$$

where K satisfies equation (3.1), M and N satisfy the following matrix equations:

$$\dot{M} + M\bar{F} + \bar{F}'M + Q_2 + PB_1'M + KB_2R_{12}^{-1}B_2'M - PR_{21}R_{11}^{-1}B_1'K + KB_1R_{11}^{-1}R_{21}R_{11}^{-1}B_1'K$$
$$+ (\bar{C} + \bar{B}P' - \bar{B}R_{11}^{-1}B_1'K)'N = 0 \qquad M(t_f) = 0 \tag{3.14}$$

$$\dot{N} + NF - T'R_{21}B_1'K + T'B_1'M + (\bar{A} + \bar{B}T')'N = 0 \qquad N(t_f) = 0 \tag{3.15}$$

and $\quad \bar{F} = A - B_1R_{11}^{-1}B_1'K - B_2R_{12}^{-1}B_2'K$

then

$$u_1 = (P' - R_{11}^{-1}B_1'K)y(t) - P'\bar{y}(t) + T'(W - \bar{W})$$
$$u_2 = -R_{12}^{-1}B_2'Kt$$

are optimal LF equilibrium pair. Furthermore, they are also team solution.

We assume that the auxiliary system (the memory system) employed by the leader is also singularly perturbed, with $w = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}$, where $w_1 \in R^{n_1}$ and $w_2 \in R^{n_2}$. Thus, the matrix functions $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ take the following forms:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \frac{\bar{A}_{21}}{\mu} & \frac{\bar{A}_{22}}{\mu} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{B}_1 \\ \frac{\bar{B}_2}{\mu} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} \\ \frac{\bar{C}_{21}}{\mu} & \frac{\bar{C}_{22}}{\mu} \end{bmatrix}$$

and

$$\bar{D} = \begin{bmatrix} \bar{D}_1 \\ \frac{\bar{D}_2}{\mu} \end{bmatrix}.$$

3.5.1. Reduced Order Games

The method, previously described, to obtain reduced order and near optimal LF strategies, with the leader using a strategy with an infinite dimensional memory can, obviously be applied to the special case

when the leader adopts strategies with only finite dimensional memory. In this section, we will use this method to obtain reduced order strategies for the case when a finite dimensional memory is employed. We will restrict ourselves to a brief description of the procedure and the results, since a detailed analysis has been given in the previous sections.

In the fast LF game, the leader designs his fast strategy as:

$$u_{1f} = -R_{11}^{-1}B_{12}'K_{1f} + P_f'(z_f - \bar{z}_f) + T_f'(w_{2f} - \bar{w}_{2f})$$

where w_{2f} satisfies:

$$\mu\dot{w}_{2f} = \bar{A}_{22}w_{2f} + \bar{C}_{22}z_f + \bar{B}_2u_{1f} + \bar{D}_2u_{2f}$$

k_{1f}, z_f are as described before and the fast subsystem and the fast part of the objective function are given in section 2.3. Sufficient conditions for the existence of the fast LF team strategies are equivalent to the ones given in proposition 3.1, with μk_{1f}, μM_f, μN_f, $\frac{B_{22}}{\mu}$, $\frac{\bar{D}_2}{\mu}$, $\frac{\bar{F}_f}{\mu}$, $\frac{\bar{B}_2}{\mu}$, Q_{23}, P_f, T_f, $\frac{\bar{C}_{22}}{\mu}$, $\frac{\bar{A}_{22}}{\mu}$, Z_f and w_{2f} replaces K, M, N, B_2, $\bar{D}_2$, $\bar{F}$, $\bar{B}_2$, Q_2, P, T, $\bar{C}$, $\bar{A}$, y and w, respectively in those equations, where

$$\bar{F}_f = (A_{22} - B_{12}R_{11}^{-1}B_{12}'K_{1f} - B_{22}R_{12}^{-1}B_2'K_{1f}).$$

In the hybrid slow game, the leader picks the following representation of his optimal slow team strategy:

$$u_{1s} = -M_{1s}x_{1s} + P_s'(x_s - \bar{x}_s) + P_f'(z_s - \bar{z}_s) + T_s'(w_{1s} - \bar{w}_{1s}) + T_f'(w_{2s} - \bar{w}_{2s}) \tag{3.16}$$

This specific representation of the leader's optimal slow team strategy has the same information structure in the limit as μ tends to zero, as the full order strategy of the leader which is described by equation (3.13).

To preserve the information structure of the full order game while solving the hybrid slow one, we need to transfer the fast information to the slow game. As before, the slow subsystem is:

$$\dot{x}_s = \hat{A}_{11}x_s + \hat{B}_{11}u_{1s} + \hat{B}_{12}u_{2s}$$

And, equivalently the slow memory subsystem can be described as:

$$\dot{w}_{1s} = \hat{\hat{A}}_{11}w_{1s} + \hat{\hat{C}}_{11}x_s + \hat{\hat{B}}_1 u_{1s} + \hat{\hat{D}}_{1s}u_{2s}$$

where

$$\hat{\hat{A}} = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21},\ \hat{\hat{C}}_{11} = \bar{C}_{11} - \bar{C}_{12}A_{22}^{-1}A_{21} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{C}_{22} + \bar{A}_{12}\bar{A}_{22}^{-1}\bar{C}_{22}A_{22}^{-1}A_{21}$$

$$\hat{\hat{B}} = \bar{B}_1 - \bar{C}_{12}A_{22}^{-1}B_{21} - \bar{A}_{12}A_{22}^{-1}\bar{B}_2 + \bar{A}_{12}\bar{A}_{22}^{-1}\bar{C}_{22}A_{22}^{-1}B_{21}$$

$$\hat{\hat{D}}_1 = \bar{D}_1 - \bar{C}_{12}A_{22}^{-1}B_{22} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2 + \bar{A}_{12}\bar{A}_{22}^{-1}\bar{C}_{22}A_{22}^{-1}B_2$$

By substituting for Z_s, $\bar{Z}_s$, w_{2s} and $\bar{w}_{2s}$, equation (3.16) can be rearranged to:

$$u_{1s} = -\tilde{M}_{1s}x_s + \tilde{P}_s'(x_s - \bar{x}_s) + \tilde{T}_s'(w_{1s} - \bar{w}_{1s}) - \tilde{L}u_{2s} - \tilde{M}\bar{x}_s \tag{3.17}$$

where the fast gains are imbeded in $\tilde{M}_{1s}$, $\tilde{P}_s$, $\tilde{T}_s$, $\tilde{L}$ and $\tilde{M}$. Now, if the leader announces the strategy (3.17), then the follower reaction can be easily determined, and the sufficient conditions for the existence of the hybrid slow LF team problem can be derived. These conditions are given in the following proposition:

<u>Proposition 3.2</u>: If there exist appropriate dimensional matrix functions $\hat{\hat{A}}_{11}(\cdot)$, $\hat{\hat{C}}_{11}(\cdot)$, $\hat{\hat{B}}_1(\cdot)$, $\hat{\hat{D}}_{1s}(\cdot)$, $P_s(\cdot)$ and $T_s(\cdot)$, so that the following holds:

$$(\tilde{R}_{22} + \tilde{L}'\tilde{R}_{21}\tilde{L} - \tilde{Q}_{23}\tilde{L})^{-1}(\tilde{Q}_{21}' - \tilde{L}'\bar{Q}_{22}^{-1} + \tilde{L}'\tilde{R}_{22}\tilde{M}_{1s} + \tilde{L}'\tilde{R}_{22}\tilde{M} - \tilde{Q}_{23}\tilde{M}_{1s} - \tilde{Q}_{23}\tilde{M}$$

$$- \tilde{L}'B_{11}'M_s + B_{12}'M_s - \tilde{L}'\hat{\hat{B}}_1'N_s + \hat{\hat{D}}_1 N_s) = M_{2s}$$

where M_s and N_s satisfy the following matrix equations

$$\dot{M}_s + M_s\bar{F}_s + \tilde{Q}_{21} - \tilde{Q}_{22}M_{2s} + \bar{Q}_{22}(-\tilde{M}_{1s} + \tilde{L}M_{2s} - \bar{M}) + (-\tilde{M}'_{1s} + \tilde{P}_s)\bar{Q}'_{22}$$

$$+ (P_s - \tilde{M}'_{1s})\tilde{R}_{21}(-\tilde{M}_{1s} + \tilde{L}M_{2s} - \tilde{M}) - (\tilde{P}_s - \tilde{M}'_{1s})(\bar{Q}'_{23}M_{2s}) + (\tilde{P}_s - \tilde{M}'_{1s})\hat{B}'_{11} + \hat{A}'_{11})M_s$$

$$+ (\hat{\bar{C}}_{11} + (\tilde{P}_s - \tilde{M}'_{1s})\hat{\bar{B}}_1)N_s = 0 \tag{3.18}$$

$$\dot{N}_s + N_s\bar{F}_s + \tilde{T}_s\bar{Q}'_{22} + \tilde{T}_s\tilde{R}_{21}(-\tilde{M}_{1s} + \tilde{L}M_{2s} - \tilde{M}) - \tilde{T}_s\bar{Q}'_{23}M_{2s} + \tilde{T}_s\hat{B}_{11}M_s$$

$$+ (\tilde{T}_s\hat{\bar{B}}_1 + \hat{\bar{A}}'_{11})N_s = 0 \tag{3.19}$$

and

$$\bar{F}_s = \hat{A}_{11} - \hat{B}_{11}M_{1s} - \hat{B}_{12}M_{2s}$$

then

$$u_{1s} = -\tilde{M}_{1s}x_s + \tilde{P}'_s(x_s - x_s) + \tilde{T}'_s(w_{1s} - \bar{w}_{1s}) - \tilde{L}u_{2s} - \tilde{M}\bar{x}_s$$

$$u_{2s} = -M_{2s}x_s$$

are the equilibrium LF hybrid slow team solution.

If the sufficient conditions for existence of the solutions of both the hybrid slow and fast LF team subgames are satisfied, then the leader can form the following reduced order strategy and apply it to the full order system.

$$u_{1ap} = u_{1c} + P'_s(x - \bar{x}_s) + P'_f(z - \bar{z}_s - \bar{z}_f) + T'_s(w_1 - \bar{w}_{1s}) + T'_f(w_2 - \bar{w}_{2s} - \bar{w}_{2f})$$

where u_{1c} (the composite control which is defined in section 3.4), P_s, P_f, T_s, T_f, $\bar{x}_s$, $\bar{z}_s$, $\bar{w}_{1s}$, $\bar{w}_{2s}$, $\bar{z}_f$ and w_{2f} are obtained from the solutions of the reduced order games. Similarly, the follower will apply $u_{2ap} = u_{2c}$, to the full order system.

Under conditions which are equivalent to the ones given in the Theorem, the low order strategies u_{1ap} and u_{2ap} are well-posed in the sense that they tend to the optimal strategies and costs respectively, in the

limit as μ tends to zero. The infinite horizon version of the problem can be easily produced by setting the derivative term in equations (3.14), (3.15), (3.18) and (3.19) equal to zero and adding the proper stabilizability conditions. It is clear that our procedure can be readily applied to infinite horizon games. Before we conclude this chapter, we present an example which shows that the sufficient conditions for the existence of the team LF slow solution may be satisfied, while these conditions are not satisfied for the full order problem.

<u>Example</u>: Let the system be described by

$$\dot{x} = -x + z$$

$$\mu\dot{z} = -z + \frac{1}{\sqrt{2}} u_1 + \frac{1}{\sqrt{2}} u_2 .$$

Let

$$J_1 = \frac{1}{2} \int_0^\infty (x^2 + z^2 + u_1^2 + u_2^2)\,dt$$

$$J_2 = \frac{1}{2} \int_0^\infty (x^2 + 2z^2 + 2u_1^2 + u_2^2)\,dt .$$

Assume player one to be the leader, while player two to be the follower.

<u>Full Order Problem</u>: The team solution is

$$\bar{u}_1 = -\frac{1}{\sqrt{2}} [0.318x + 0.414z] = -0.225x - 0.293z$$

$$\bar{u}_2 = -\frac{1}{\sqrt{2}} [0.318x + 0.414z] = -0.225x - 0.293z$$

and the leader's cost $\equiv \frac{1}{2} y'(0)Sy(0)$ where $y = \begin{bmatrix} x \\ z \end{bmatrix}$ and

$$S = \begin{bmatrix} 0.450 & 0.318 \\ 0.318 & 0.414 \end{bmatrix} .$$

Let the leader choose his strategy to be of the form

$$u_1(t) = -0.225x(t) + P_1(x(t)-\bar{x}(t)) - 0.293z + P_2(z-\bar{z}(t))$$

where $(\bar{x}, \bar{z})$ is the optimal trajectory resulting from applying $\bar{u}_1(t)$ and $\bar{u}_2(t)$.

If the leader announces the above strategy, then the optimal reaction of the follower will be unique and a function of P_1 and P_2. It can be checked that there exist no P_1 and P_2 by which the leader can force the follower to play with him as a team. Clearly, if we use our procedure, which depends on preserving the information structure of the full order problem, to solve the example, we will find that the leader cannot enforce his team solution by adopting the aforementioned affine strategies. Instead, let us solve the pure slow (no information about the fast game is used) problem.

<u>The Slow Problem</u>

$$\dot{x}_s = -x_s + \frac{1}{\sqrt{2}}(u_{1s} + u_{2s})$$

$$J_{1s} = \frac{1}{2}\int_0^\infty (x_s^2 + 1.5u_{1s}^2 + 1.5u_{2s}^2 + u_{1s}u_{2s})dt$$

$$J_{2s} = \frac{1}{2}\int_0^\infty (x_s^2 + 3u_{1s}^2 + 2u_{2s}^2 + 2u_{1s}u_{2s})dt.$$

<u>Team Solution</u>:

$$\bar{u}_{1s} = -0.159x_s, \qquad \bar{u}_{2s} = -0.159x_s.$$

Assume the leader chooses

$$\tilde{u}_{1s} = -0.159x_s + \lambda(x_s - \bar{x}_s).$$

By solving the problem, we can see that if the leader chooses $\lambda = -3$, the

sufficient conditions for existence of a team leader-follower for the slow problem is satisfied. So without preserving the information structure, the sufficient conditions of the reduced order problem may not correspond to those of the full order problem. □

The procedure given in this chapter which is based on preservation of information structure of the full order problem while solving the reduced order ones can be applied in general to all linear quadratic singularly perturbed games. So, we expect a well-posed solution of the usual reduced order problem if the optimal strategies do not depend on the information structure as it is the case for control problems and zero-sum games.

PART TWO

NASH AND LEADER-FOLLOWER STRATEGIES WITH DECISION-DEPENDENT INFORMATION STRUCTURE

CHAPTER 4

NASH STRATEGIES WITH DECISION-DEPENDENT INFORMATION STRUCTURE AND A NEW MODEL OF DUOPOLY

4.1. Introduction

In this chapter we consider Nash games with a decision-dependent information structure (DDIS) whereby one player formulates his strategy as a function of the decision of the other. We compare properties of the solutions to those involving a normal information structure (NIS) whereby the strategy of each player is not a function of the decision of the other. In Section 4.2, we discuss the equilibrium Nash solution with two types of information structure, we compare the Stackelberg solution concept with the Nash solution with DDIS, and we give two examples which clarify the basic ideas in this section. In the first part of Section 4.3, we consider a general static market model of duopoly and derive the necessary conditions for the supply adjustment controls of both firms to be optimal in the Nash sense with the two types of information structure. In the second part of Section 4.3, we analyze in detail the case of linear market demand relation and quadratic cost function, we obtain analytic solutions for the optimal strategies of the two firms, we show that the profit of the firm with DDIS is more than its corresponding profit with NIS, and we discuss the general properties in terms of marginal cost, marginal revenue, price, and the consumer's welfare. In Section 4.4, we generalize the concept of DDIS to multistage dynamic games, we solve a two-stage dynamic duopoly with a linear demand and quadratic cost function, and we give sufficient conditions for existence of the Nash equilibrium solution with DDIS for the discrete linear quadratic problem.

4.2. Nash Solution

Let Γ_1 and Γ_2 be the spaces of admissible strategies for players one and two (P_1 and P_2) respectively, with $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. The strategies γ_i are mappings from the information space to the control action space of each player. The nature of the information structure should be specified in each situation. Let $J_1(\gamma_1,\gamma_2)$ and $J_2(\gamma_1,\gamma_2)$ be the corresponding payoff functions of the two players. A Nash equilibrium assumes that if one player maximizes on the basis that the other player's strategy is known and it is at equilibrium, then this player will find his optimal strategy at equilibrium.

Definition : The rational reaction set of player i to permissible strategies of player j, $D^i(\gamma_j)$ is

$$D^i(\gamma_j) = \{\gamma_i^* \in \Gamma_i \text{ such that } J_i(\gamma_i^*,\gamma_j) \geq J_i(\gamma_i,\gamma_j) \text{ for all } \gamma_i \in \Gamma_i \text{ and each } \gamma_j\}.$$

If the reaction strategy of player i against γ_j is unique, then we can describe the reaction set $D^i(\gamma_j)$ by $f_i(\gamma_j)$ where $\gamma_i = f_i(\gamma_j)$.

4.2.1. Normal vs. Decision-Dependent Information Structure (NIS vs. DDIS)

In a Nash game both players are required to declare their strategies simultaneously before the start of the game, but the sequence of their action depends on further rules and assumptions of the game. If both players apply their strategies simultaneously, the decision of any of the players will not be available to the other, and as a result no player can formulate his strategy as a function of the other player's decision, and their role in the game is totally symmetric. In such a case we say that the information structure

of the game is normal. If the two players[1] apply their strategies sequentially, let us assume that P_2 plays first, P_1 has access to the decision of P_2, and P_1 has a choice of using this information in his strategy. Thus P_1 may declare a strategy which is a function of P_2's decision, but the declarations are simultaneous. The difference between normal and decision-dependent information structure in Nash games will be clarified in the following example.

Example 4.1:Let $J_1(u_1,u_2)$ and $J_2(u_1,u_2)$ be the payoff functions of P_1 and P_2 respectively, where

$$J_1(u_1,u_2) = -u_1^2 - 2u_2^2 - 3u_1$$

$$J_2(u_1,u_2) = -3u_2^2 - 10u_2 - 2u_1u_2.$$

Under the normal information structure assumption, no player can formulate his strategy as a function of the decision of the other player. So each player will maximize his objective function on the basis that the strategy of the other player is fixed, yielding

$$u_{1N} = -3/2 \quad \text{and} \quad u_{2N} = -7/6$$

where (u_{1N},u_{2N}) constitutes the Nash equilibrium solution with a normal information structure. The Nash values of the payoff functions are $J_{1N} = -0.472$, $J_{2N} = +4.087$.

If P_2 will play first, and the value of u_2 will be available to P_1 when he applies his strategy, P_1 can announce a strategy which is a function

[1]A two-player Nash game with decision-dependent information structure can be generalized to N-player as follows: Let us arrange the N players in such a way as P_1 makes the first move, P_2 the second move,..., and P_N the last move. So each player i can formulate his strategy as a function of $P_1,P_2,\ldots,P_{i-1}$ decisions for all $i = 2,\ldots,N$.

of u_2. For simplicity, let this function be restricted to be linear in u_2, i.e., $u_1 = bu_2$.

The Nash equilibrium solution with DDIS can be determined as follows: P_1 maximizes J_1 on the assumption that u_2 is fixed, and we have

$$u_{1D} = -\frac{3}{2}.$$

P_2 maximizes J_2 on the assumption that $u_1 = bu_2$, and the optimal u_2 is

$$u_{2D} = \frac{-5}{3+2b}.$$

It can be seen that if there exists a real number b such that

$$bu_{2D} = -\frac{3}{2}$$

then the Nash solution with DDIS exists. Substituting for u_{2D} in the above equations, and solving for b, we find that $b = 2.25$ is the unique solution. Thus, there is a unique equilibrium Nash solution with DDIS, which is

$$(2.25u_{2D}, -0.667).$$

The values of the payoff functions which correspond to this equilibrium solution are $J_{1D} = 1.3602$ and $J_{2D} = +3.334$. P_1 has benefited from his additional information (the decision of P_2), since his payoff function under DDIS is greater than the corresponding payoff function under NIS. P_2 suffers a little but the collective payoff, $J_1 + J_2$ is improved.

4.2.2. Nash Solution with DDIS vs. Stackelberg Solution

The Stackelberg solution assumes that the two players have different roles. There is a leader and there is a follower. The leader announces his

strategy first[2] and as a result he can impose a solution which is favorable to himself. It is the order of announcing, not the order of action, that distinguishes the leader from the follower.

As was pointed out, in the Nash solution with DDIS, both players announce their strategies simultaneously, where P_1 announces a strategy which is a function of P_2's decision, i.e., $u_1 = g_1(u_2)$. If P_1 chooses g_1 to be the same as f_1 (see the previous Definition), then P_1 is choosing a strategy which is the same as his reaction function. It is not difficult to see that the best strategy for P_2 in the Nash sense will be equivalent to the Stackelberg equilibrium solution with P_2 as the leader and P_1 as the follower, and with NIS, that is, the leader plays first. Such a choice of g_1 does not worsen P_2's payoff function (it may improve it) since the payoff function of the leader in the Stackelberg solution is at least as good as (and possibly better than) that of the corresponding Nash solution with NIS (see [5]). But the value of P_1's payoff function may be worse. If the follower applies his strategy first, then the leader can formulate a strategy which is a function of the follower's decision. This enhances the leader's opportunity to enforce a team solution. (For a discussion of the Stackelberg team solution, see [26]).

Determination of the Nash solution with DDIS and thus the form of the mapping g_1 which achieves the best J_1, is not an easy problem and it needs further investigation. But it is desirable for P_1 to make use of the information available to him (the strategy of P_2) by choosing an appropriate function g_1, which increases his payoff relative to the corresponding Nash payoff with NIS. Clearly, P_1 can always disregard the information available to him and choose a robust strategy (a strategy which is insensitive to any decision made by the other player). The solution in this case will be the Nash solution with NIS.

[2]The leader can announce his strategy first either due to the lack of information of the other player about the leader's performance index or due to differences in size and strength.

A simple example which illustrates some of the basic properties of the solution is presented below.

Example 4.2: The payoff functions J_1 and J_2, defined on $R^1 \times R^1$, are assumed to be convex and have contour lines as shown in Figure 4.1. The reaction curves $D^1(u_2)$ and $D^2(u_1)$ are loci of tangent points between the contour lines and the lines of constant u_1 and u_2, respectively. Point N, whose coordinates are (u_{1N}, u_{2N}) is the Nash equilibrium solution with normal information structure. Point S is the Stackelberg solution with P_2 as a leader and P_1 as a follower under NIS. $(g_1(u_{2D}), u_{2D})$ is the resulting Nash equilibrium solution, with P_1's strategy depending on P_2's decision. This equilibrium point is evaluated as follows: P_1 finds his reaction set $D^1(u_2)$. P_2 maximizes $J_2(u_1,u_2)$ taking the declared strategy of P_1, which is $u_1 = g_1(u_2)$, into account and finds u_{2D} as the optimal solution. If $D^1(u_{2D}) = g_1(u_{2D})$, as it is in Figure 4.1, then the Nash solution with DDIS structure exists.

4.3. Market Model of Duopoly

Let there be two firms which have access to the same potential buyers. The two firms share the production of a commodity (or two perfect substitute commodities) with the quantity of production of each firm as its strategic variable. The market price is determined by a special demand mechanism. In this demand mechanism, the market is cleared of whatever quantities the firms offer. The sales are assumed to be made on one occasion; thus, actions for a sequence of periods are ruled out. The Nash equilibrium solution assumes that if one duopolist maximizes on the basis that the other duopolist's output is known and it is at equilibrium, then this duopolist will produce the equilibrium output.

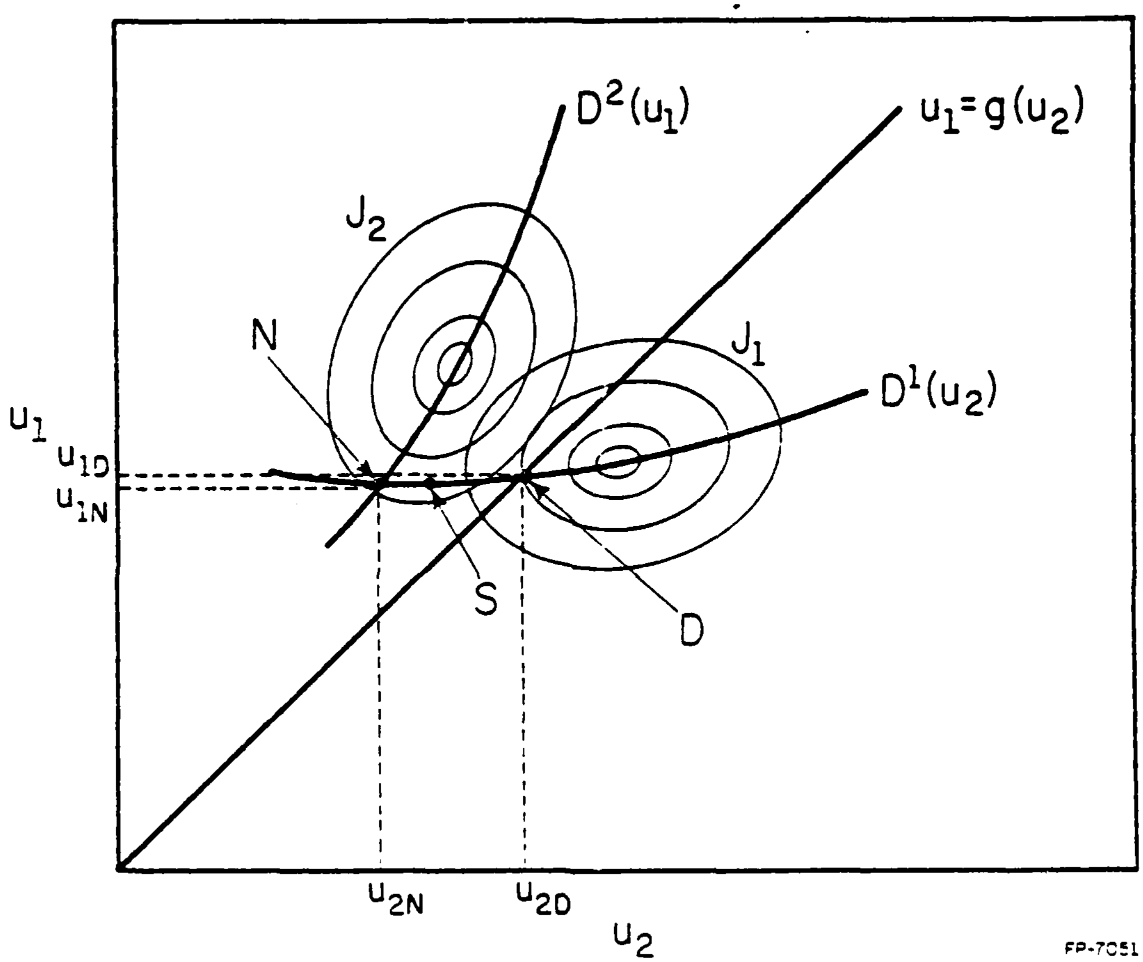

Figure 4.1. A game with the two types of Nash solution and with Stackelberg solution.

4.3.1. General Model

Let q_i denote the output rate produced by firm i and Q denote the total marketed quantity. Hence.

$$Q = q_1 + q_2.$$

Let us suppose that the aggregate demand can be represented by a continuous function as follows:

$$p = f(Q)$$

such that the price p and the quantity Q vary inversely. It is reasonable to assume that each firm wishes to maximize its net profit J_i, where

$$J_i = pq_i - h_i(q_i)$$

where $h_i(q_i)$ is what it costs firm i to produce at a rate q_i, and it is assumed to be an increasing function of q_i.

4.3.1.1. Nash Equilibrium Solution with NIS

With NIS no firm can formulate its strategy as a function of the other firm's decision. Nash theory postulates that each firm chooses a rate of sale that maximizes its net revenue given the competitor's rate of sale. The equilibrium Nash solution with NIS can be found by solving the two equations[3]

$$\frac{\partial J_i(q_1,q_2)}{\partial q_i} = f(Q) + \frac{\partial f(Q)}{\partial Q} q_i - \frac{\partial h_i(q_i)}{\partial q_i} = 0 \quad \text{for } i = 1,2.$$

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[3]We require that $q_i \geq 0$, so the necessary conditions should be written as an inequality as follows: $\frac{\partial J_i(q_1,q_2)}{\partial q_i} \leq 0$. But since this is not the main issue of the paper, we assume that the solution obtained is nonnegative, for simplicity.

It is clear that firm i should choose its rate of sale such that its marginal cost (MC) equals its marginal revenue (MR). In addition, the price (p) is higher than the marginal cost (MC) (since $\frac{\partial f(Q)}{\partial Q} < 0$), a situation which is different from perfect competition where price (p) equals marginal cost (MC).

4.3.1.2. Nash Equilibrium Solution with DDIS

Whereas the two firms announce their strategies simultaneously as with NIS, the amount supplied by firm two will be known when firm one supplies its commodity, and firm one is going to use this information structure in its declared strategy. Such a type of information structure can be realized if firm one has spies or agents, who inform their management of the decisions of firm two, or if firm two has to apply its strategy first (e.g., the plant of firm two may be at a distance which is farther from the market than that of firm one, so to compensate for the effect of transportation delay, firm two may have a faster production facility). Due to the information structure of the game, the strategy space of firm two can be described as $q_2 = a_2$, where $a_2 \in R^+$; the strategy space of firm one can be described by $q_1 = g_1(q_2)$, where the function g_1 is chosen by firm one.

The procedure to find the Nash equilibrium solution with DDIS is as follows:

(1) Firm one will maximize its net profit on the assumption that $q_2 = a_2$, which can be put mathematically as

$$\frac{\partial J_1(q_1,q_2)}{\partial q_1} = f(Q) + \frac{\partial f(Q)}{\partial Q} q_1 - \frac{\partial h_1(q_1)}{\partial q_1} = 0.$$

Solving for q_1 as a function of q_2 in the above equation, firm one obtains its reaction function $f_1(\cdot)$ and $q_1 = f_1(q_2)$.

(2) Firm two will maximize its net profit on the assumption that $q_1 = g_1(q_2)$ and find q_{2D} as the optimal output rate as follows

$$f(Q) + \frac{\partial f(Q)}{\partial Q} \frac{\partial Q}{\partial q_2} q_2 - \frac{\partial h_2(q_2)}{\partial q_2} = 0.$$

In this case $\frac{\partial Q}{\partial q_2} \neq 1$ as it is for NIS, but

$$\frac{\partial Q}{\partial q_2} = 1 + \frac{\partial g_1(q_2)}{\partial q_2} .$$

It can be seen that depending on the value of $\frac{\partial g_1(q_2)}{\partial q_2}$ at the optimal strategy, we may have the marginal cost (MC) either higher, equal, or lower than the price (p).

(3) If $g_1(q_{2D}) = f_1(q_{2D})$, then the Nash solution with DDIS exists.

4.3.2. A Linear Model Case

In this section, we will seek analytic solutions for the Nash game with NIS and DDIS. This will help us understand the difference between the two solution concepts and find the extra profit which firm one may have obtained due to its extra information.

Let Q and p satisfy an inverse linear[4] demand relation given as

$$Q = d_o + d_1 p$$

where d_o is a positive constant, while d_1 is a negative constant. Let the cost function $h_i(q_i)$ be of the form

$$h_i(q_i) = \frac{1}{2} c_i q_i^2 + k_i$$

where k_i and c_i are positive constants.

4.3.2.1. Nash Equilibrium Solution with NIS

Firm i maximizes J_i over q_i under the constraint that q_j is given, where $i,j=1,2$ and $i \neq j$. The reaction curve of firm i can be found to be

$$q_i = \frac{d_o}{2-c_i d_1} - \frac{1}{2-c_i d_1} q_j .$$

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[4] See [36] for a discussion of a linear demand relation.

Solving the above equations for $i=1,2$ simultaneously, we find that the Nash equilibrium solution with NIS is

$$q_{1N} = \frac{d_o - c_2 d_1 d_o}{3 - 2c_1 d_1 - 2c_2 d_1 + c_1 c_2 d_1^2}$$

$$q_{2N} = \frac{d_o - c_1 d_1 d_o}{3 - 2c_1 d_1 - 2c_2 d_1 + c_1 c_2 d_1^2}.$$

Thus (q_{1N}, q_{2N}) is a feasible solution since the q_{iN} are clearly non-negative.

4.3.2.2. Nash Equilibrium Solution with DDIS

For simplicity, let us assume that the space of admissible strategies of firm one is linear in the decisions of firm two, i.e., $q_1 = bq_2$, where $b \in R$ Firm one will maximize J_1 on the basis that q_2 is fixed, and $p = \frac{q_1 + q_2 - d_o}{d_1}$, and it will find q_{1D} to be

$$q_{1D} = \frac{d_o}{2 - c_1 d_1} - \frac{1}{2 - c_1 d_1} q_{2D}.$$

Firm two will maximize J_2 on the basis that $q_1 = bq_2$ and $p = \frac{q_1 + q_2 - d_o}{d_o}$ and it obtains

$$q_{2D} = \frac{d_o}{2 + 2b - c_2 d_1}$$

which is clearly in the strategy space of firm two. A sufficient condition for (q_{1D}, q_{2D}) to be an equilibrium Nash pair with DDIS is that

$$bq_{2D} = \frac{d_o}{2 - c_1 d_1} - \frac{1}{2 - c_1 d_1} q_{2D}.$$

Substituting for q_{2D} in the above equation and solving for b, we find that

$$b = \left(\frac{1 - c_2 d_1}{-c_1 d_1}\right).$$

Hence, (q_{1D}, q_{2D}) constitutes a unique equilibrium Nash pair with DDIS, where

$$q_{1D} = \left(\frac{1 - c_2 d_1}{-c_1 d_1}\right) q_{2D}$$

$$q_{2D} = \frac{-c_1 d_o d_1}{2 - 2c_1 d_1 - 2c_2 d_1 + c_1 c_2 d_1^2}.$$

It can be easily shown that (q_{1D}, q_{2D}) is a feasible solution (i.e., the output rate of each of the firms is non-negative).

The question which remains to be answered is whether firm one benefits from the additional information it has available (the output rate of firm two). This question will be answered in the following proposition.

Proposition 4.1: In a market duopoly with a linear DDIS, linear demand relation and quadratic cost functions, the profit of the firm possessing the additional decision information is increased compared to that in the absence of information. Furthermore, the profit of the other firm is decreased compared to its profit in NIS.

Proof: The net profit of firm one, J_1, is as follows

$$J_1 = pq_1 - \frac{1}{2} c_1 q_1^2 - k_1.$$

Substituting for p by $\left(\frac{q_1 + q_2 - d_o}{d_1}\right)$, J_1 becomes

$$J_1 = q_1^2 \left(\frac{2 - c_1 d_1}{2d_1}\right) + \frac{q_1 q_2}{d_1} - \frac{q_1 d_o}{d_1} - k_1.$$

The optimal strategy of firm one in both NIS and DDIS should satisfy its reaction curve equation, namely,

$$q_1 = \frac{d_o}{2 - c_1 d_1} - \frac{1}{2 - c_1 d_1} q_2.$$

Hence, J_1 on the reaction curve of firm one, $\bar{J}_1$, is as follows

$$\bar{J}_1 = q_1^2\left(\frac{2 - c_1d_1}{-2d_1}\right) - k_1$$

$\bar{J}_1$ is a strictly increasing function in q_1. So it is sufficient to show that $q_{1D} > q_{1N}$ to prove the first part of the proposition. But this is the case since

$$q_{1D} = \frac{d_o - c_2d_1d_o}{2 - 2c_1d_1 - 2c_2d_1 + c_1c_2d_1^2}$$

which is obviously greater than q_{1N}. For firm 2, the profit under NIS (J_{2N}) is

$$J_{2N} = q_{2N}^2\left(\frac{2-c_2d_1}{-2d_1}\right) - k_2 .$$

The profit under DDIS (J_{2D}) is

$$J_{2D} = q_{2D}\left(\frac{-d_o}{2d_1}\right) - k_2 .$$

If $J_{2D} \geq J_{2N}$, then we have

$$\frac{-c_1d_1d_o^2}{a} - (2-c_2d_1)d_o^2\left(\frac{1-c_1d_1}{a+1}\right)^2 \geq 0$$

where

$$a = 2 - 2c_1d_1 - 2c_2d_1 + c_1c_2d_1^2 .$$

After some manipulation, we find that

$$-2 + c_1d_1 + 2c_2d_1 - c_1c_2d_1^2 \geq 0$$

which is a contradiction, so $J_{2N} > J_{2D}$. ■

Consequently, firm two should not let its output rate of production be known to firm one, because it will end up suffering.

Before concluding this section, several important remarks should be mentioned.

R4.1: In the special case when the two firms have the same cost function ($c_1 = c_2 = c$), the two firms will have the same output rate, and as a result, the same net profit of Nash game with NIS as the solution concept. But it can be shown that the net profit of firm one is more than that of firm two under DDIS.

R4.2: If we compare the total output of the two firms under NIS and DDIS, we find that

$$Q_N = q_{1N} + q_{2N} = \frac{2d_o - c_1 d_1 d_o - c_2 d_1 d_o}{3 - 2c_1 d_1 - 2c_2 d_1 + c_1 c_2 d_1^2}$$

and

$$Q_D = q_{1D} + q_{2D} = \frac{d_o - c_2 d_1 d_o - c_1 d_1 d_o}{2 - 2c_1 d_1 - 2c_2 d_1 + c_1 c_2 d_1^2} .$$

It can be easily checked that the total output rate of the two firms under DDIS will be less than the total output rate under NIS. This implies that the welfare[5] of the consumers is worse off under DDIS conditions, since the firms sell less quantity at higher prices in these conditions.

R4.3: The reaction curve of firm i under NIS can be expressed as follows

$$p_N(q_{1N}, q_{2N}) - c_i q_{iN} = \frac{-1}{d_1} q_{iN} \qquad \text{for } i = 1,2.$$

Under DDIS the reaction curve of firms one and two can be expressed, respectively, as follows

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[5] An appropriate definition of consumers social welfare $W(Q)$ is the consumer's surplus, which is mathematically given as $W(Q_N) = \int_0^{Q_N} (P(s) - P_N)$, where Q_N and P_N are the equilibrium values of quantity and price, respectively.

$$p_D(q_{1D},q_{2D}) - c_1 q_{1D} = \frac{-1}{d_1} q_{1D} \qquad \text{for firm one,}$$

and

$$p_D(q_{1D},q_{2D}) - c_2 q_{2D} = \frac{-1}{d_1} (1+b) q_{2D} \qquad \text{for firm two,}$$

where

$$b = \left(\frac{1 - c_2 d_1}{-c_1 d_1}\right).$$

From the above equations, it is seen that the price (p) is higher than the marginal cost for both firms and for the two kinds of information structures. But for each firm the difference between the price (p) and the marginal cost (MC) is higher for the solution with DDIS than for NIS. Hence, Nash solutions with NIS and with DDIS do not satisfy group rationality.[6] But the Nash solution with NIS is closer to the "perfect competition" solution than that with DDIS since the difference between price (p) and marginal cost (MC) is less for each firm. ·

R4.4: In the linear duopoly model we examined, we assumed the strategies of firm one to be of the form $q_1 = bq_2$. Without this assumption the DDIS Nash condition yields a general class of nonunique nonlinear solutions $q_1 = g(q_2)$, which have values of (bq_{2D}) and slopes equal to (b) at $q = q_{2D}$. Consequently, we will have a class of equilibrium Nash solutions which leads to the same payoff functions but with different sensitivity properties.

R4.5: If the strategy space of firm one is assumed to be of the form $q_1 = a_1 + b_1 q_2$, where a_1 and $b_1 \in R$; while keeping the strategy space of firm two unchanged, the reaction curve of firm one will stay as before, i.e.

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[6] To satisfy group rationality, which includes the buyers, the price must equal the marginal cost (see [34]).

$$q_{1D} = \frac{d_o}{2 - c_1 d_1} - \frac{1}{2 - c_1 d_1} q_{2D}.$$

But the reaction curve of firm two will be

$$q_{2D} = \frac{d_o - a_1}{2 + 2b_1 - c_2 d_1} .$$

It can be seen that depending on the values of a_1 and b_1, we have different equilibrium Nash solutions, hence we have an uncountably infinite number of Nash solutions with DDIS (for a more detailed discussion of a similar situation, the reader is referred to [24]). These equilibrium Nash strategies are not implementable since neither firm knows which Nash strategies the other firm will apply.

R4.6: The price elasticity of demand at the Nash equilibrium point with NIS(η_N) is

$$\eta_N = \frac{1 - c_1 d_1 - c_2 d_1 + c_1 c_2 d_1^2}{2 - c_1 d_1 - c_2 d_1} .$$

The price elasticity of demand at the Nash equilibrium point with DDIS is

$$\eta_D = \frac{1 - c_1 d_1 - c_2 d_1 + c_1 c_2 d_1^2}{1 - c_1 d_1 - c_2 d_1} .$$

It can be seen that $\eta_D > \eta_N$, and the market is always price elastic (an increase (decrease) in price leads to a reduction (increase) in the amount of money spent on the commodity) under the conditions of DDIS but under NIS conditions it will be price elastic if and only if $d_1^2 c_1 c_2 > 1$.

R4.7: The duopoly model is necessarily simplistic to avoid technical complications. However, it serves to clarify certain aspects of the nature of duopolist competition. Furthermore, this motivates examination of various forms of strategy spaces from a game theoretic viewpoint.

4.4. Multistage Dynamic Problem

In this section we will generalize the basic idea presented in Section 4.2 to multistage dynamic games. Let the evolution of the system be described by

$$x_{k+1} = h_k(x_k, u_{1k}, u_{2k}, k)$$

where $x_k \in X^{n_k}$ is the state of the game at stage k, $u_{ik} \in U^{m_{ik}}$ is the control of P_i at stage k for i=1,2. The function h_k is continuously differentiable in x_k, u_{1k}, u_{2k}. Let the objective function of P_i be

$$J_i(u_1, u_2) = S_N^i(x_N) + \sum_{k=0}^{N-1} S_k^i(x_k, u_{1k}, u_{2k})$$

where

$$u_i = (u_{io}, \ldots, u_{i_{N-1}})$$

and the function S_k^i is continuously differentiable in x_k, u_{1k}, u_{2k}.

Let us assume that at the start of each stage of the game, P_2 makes the first move and P_1 makes the second move. So the information which may be (but not necessarily) available to player i, at stage k, let us call it $\eta_i(k)$, is

$$\eta_2(k) = \{x_o, x_1, \ldots, x_k; \quad u_{20}, \ldots, u_{2k-1}, u_{10}, \ldots, u_{1k-1}\}$$

$$\eta_1(k) = \{x_o, x_1, \ldots, x_k; \quad u_{10}, \ldots, u_{1k-1}, u_{20}, \ldots, u_{2k}\}.$$

If the information available to the players is memoryless ($\eta_2(k) = x_k$ and $\eta_1(k) = (x_k, u_{2k})$), then P_1 and P_2 have to choose $u_{1k} = \gamma_{1k}(x_k, u_{2k})$ and $u_{2k} = \gamma_{2k}(x_k)$ respectively, as their optimal strategies in the Nash sense.

As in the static case, the general solution is extremely difficult to obtain. But if we assume a certain strategy form for P_1 and P_2, solve a control optimization problem for each player on the assumption that the strategy

form of the other player is given, and if the resulting optimal strategies have the same form as our assumed ones, then we say that such a strategy pair is a Nash equilibrium solution with decision-dependent information structure. If the resulting optimal strategies do not have the same form as our assumed ones, then we try to find sufficient conditions for the equality of the two forms, and these conditions will be sufficient for the existence of a Nash equilibrium solution with the assumed strategy forms. Before we give the sufficient conditions for existence of a Nash solution with DDIS, we will look into the problem of a two-stage dynamic duopoly.

4.4.1. Two-Stage Dynamic Duopoly

Assume that the two duopolists face a demand relation of the form

$$Q_t = d_o + d_1 p_i + d_2 p_{t+1}$$

where

$$Q_t \equiv \text{total quantity demanded in period } t$$

$$p_t \equiv \text{price in period } t$$

and $d_o > 0$, $d_1 < 0$, and $d_2 > 0$.

Such a demand relation is more realistic than the static one since the quantity demanded by the consumers in period t depends not only on current, but also on the buyer's view of future prices, (see [37]), since if the future price is going to be high, the consumers will have high current demand. Let us assume that two firms produce the same commodity (or two perfect substitute commodities), and share the same market. Let $q_t^{(1)}, q_t^{(2)}$ be the quantities supplied by the two respectively, in period t, so the demand equation will be

$$q_t^{(1)} + q_t^{(2)} = d_o + d_1 p_t + d_2 p_{t+1}.$$

period t $(r_t^{(i)})$ is

$$r_t^{(i)} = p_t q_t^{(i)} - C_t^{(i)}$$

where $C_t^{(i)}$ is the cost of producing $q_t^{(i)}$, which is assumed to be quadratic in $q_t^{(i)}$ (i.e. $C_t^{(i)} = \frac{1}{2} c_t^{(i)} q_t^{2(i)}$). The present value of the total profit of each firm is

$$(PV)_i = \sum_{t=0}^{1} \beta^t r_t^{(i)}$$

where $\beta \equiv$ discount factor. We assume $\beta = 1$. Thus the payoff function of firm i can be expressed as

$$(PV)_i = \sum_{t=0}^{1} p_t q_t^{(i)} - \frac{1}{2} c_t^{(i)} q_t^{2(i)}.$$

4.4.1.1. Cournot-Nash Solution with Normal Information Structure

If the information structure of both firms is normal, then at the start of each stage, both firms supply the market with their product simultaneously, and as a result no one firm can declare a strategy which is a function of the decision of the other one.

Using dynamic programming and the definition of Nash solution we find that

At stage 1: $q_{1N}^{(1)} = \dfrac{p_1}{c_1^{(1)}}$, $q_{1N}^{(2)} = \dfrac{p_1}{c_1^{(2)}}$.

At stage 0: The reaction curve for firm one is

$$q_o^{(1)} = \frac{d_o - q_o^{(2)}}{1 - d_2^2 c_1^{(1)} c_o^{(1)}} + p_o \left[\frac{d_1 - c_1^{(1)} d_2^2}{1 - d_2^2 c_o^{(1)} c_1^{(1)}} \right]$$

and the reaction curve of firm two is

$$q_o^{(2)} = \frac{d_o - q_o^{(1)}}{1 - d_2^2 c_1^{(2)} c_o^{(2)}} + p_o \left[\frac{d_1 - c_1^{(2)} d_2^2}{1 - d_2^2 c_o^{(2)} c_1^{(2)}} \right].$$

Solving the above equations simultaneously, we have

$$q_{oN}^{(1)} = \frac{-d_o c_o^{(2)} c_1^{(2)} + p_o[-d_1 c_o^{(2)} c_1^{(2)} - c_1^{(1)} + d_2^2 c_1^{(1)} c_1^{(2)} c_o^{(2)} + c_1^{(2)}]}{d_2^2 c_o^{(1)} c_1^{(1)} c_o^{(2)} c_1^{(2)} - c_o^{(1)} c_1^{(1)} - c_o^{(2)} c_1^{(2)}}$$

$$q_{oN}^{(2)} = \frac{-d_o c_o^{(1)} c_1^{(1)} + p_o[-d_1 c_o^{(1)} c_1^{(1)} - c_1^{(2)} + c_1^{(1)} + d_2^2 c_1^{(2)} c_1^{(1)} c_o^{(1)}]}{d_2^2 c_o^{(1)} c_1^{(1)} c_o^{(2)} c_1^{(2)} - c_o^{(1)} c_1^{(1)} - c_o^{(2)} c_1^{(2)}} .$$

In order that $(q_{oN}^{(1)}, q_{1N}^{(1)}, q_{oN}^{(2)}, q_{1N}^{(2)})$ be a feasible Nash solution, we should have $q_{oN}^{(1)}, q_{1N}^{(1)}, q_{oN}^{(2)}$, and $q_{2N}^{(2)}$ nonnegative. Sufficient conditions for the feasibility of the equilibrium Nash solution can be easily found.

4.4.1.2. Nash Solution with DDIS

If the information structure of firm one is decision-dependent, then at the start of each stage, firm two supplies the market with its product before firm one, and as a result firm one may declare a strategy which is a function of the decision of firm two.

At stage 1: $q_{1D}^{(1)} = \dfrac{p_1}{c_1^{(1)}}$, $\quad q_{1D}^{(2)} = \dfrac{p_1}{c_1^{(2)}}$.

At stage 1 the decision of any firm does not affect the profit of the other firm at this stage, so it is meaningless to have a strategy which depends on the current decision of the other firm, but we can have a strategy which depends on the past decisions of the other firm.

At stage 0: For any strategy form of firm two, firm one will respond by its reaction curve

$$q_o^{(1)} = \frac{d_o - q_o^{(2)}}{1 - c_o^{(1)} c_1^{(1)} d_2^2} + p_o \left[\frac{d_1 - c_1^{(1)} d_2^2}{1 - c_o^{(1)} c_1^{(1)} d_2^2} \right] .$$

Let us assume that the space of admissible strategies of firm one at this stage is linear in the current decision of firm two. Hence firm one will declare a strategy of the form

$$q_o^{(1)} = bq_o^{(2)}.$$

The optimal response of firm two will be

$$q_{oD}^{(2)} = \frac{p_o(-d_2^2c_1^{(2)} + d_1(1+b)) + d_o(1+b)}{(1+b)^2 - c_o^{(2)}d_2^2c_1^{(2)}}.$$

For an equilibrium Nash solution with DDIS to exist, firm one should have

$$bq_{oD}^{(2)} = \frac{d_o - q_{oD}^{(2)}}{1-c_o^{(1)}c_1^{(1)}d_2^2} + p_o\left[\frac{d_1 - c_1^{(1)}d_2^2}{1-c_o^{(1)}c_1^{(1)}d_2^2}\right].$$

If we substitute for $q_{oD}^{(2)}$ in the above equation and solve for b, we find that b should satisfy the following algebraic equation

$$Ab^2 + Bb + C = 0$$

where

$$A = -d_oc_o^{(1)}c_1^{(1)}d_2^2 - d_1p_oc_o^{(1)}c_1^{(1)}d_2^2 + p_oc_1^{(1)}d_2^2$$

$$B = -p_od_2^2c_1^{(2)} - c_o^{(1)}c_1^{(1)}d_2^2d_o - c_o^{(1)}c_1^{(1)}d_2^2p_o(d_1 - d_2^2c_1^{(2)}) + 2p_oc_1^{(1)}d_2^2$$

$$C = -p_od_2^2c_1^{(2)} + p_od_2^2c_1^{(1)} + c_o^{(2)}c_1^{(2)}d_2^2p_o(d_1 - c_1^{(1)}d_2^2) + d_2^2d_oc_o^{(2)}c_1^{(2)}.$$

Consequently, we may have a unique solution, two solutions, or no solution which satisfy the Nash rationale with DDIS. A comparison between the payoff functions under NIS and DDIS can be made for firm one, but due to manipulational difficulties it will not be pursued here.

It is our conjecture that if a unique equilibrium Nash solution with DDIS exists, then in a two-stage dynamic duopoly, with linear DDIS, linear demand relation and quadratic cost functions, the profit of the firm possessing the additional decision information will be increased compared to that in the absence of information. We failed to find a counterexample to our guess and the following numerical example is illustrative.

Example 4.3: Let the market demand relation be

$$q_t^{(1)} + q_t^{(2)} = 6 - 2p_t + p_{t+1}; \qquad p_o = 2.$$

Let us assume that the two firms have identical cost functions with

$$c_o^{(1)} = c_o^{(2)} = 1; \qquad c_1^{(1)} = c_1^{(2)} = 3.$$

Under NIS, the Nash output rate value is

At stage 0: $q_{oN}^{(1)} = q_{oN}^{(2)} = 4.$

At stage 1: $q_{1N}^{(1)} = q_{1N}^{(2)} = 2.$

The resulting Nash profit of each firm (PV_N) is

$$(PV_N)_1 = (PV_N)_2 = 6.$$

Under DDIS, with firm one declaring a decision-dependent strategy, the Nash output rate value is

At stage 0: $q_{oD}^{(2)} = 12.057; \qquad q_{oD}^{(1)} = \frac{2}{3} q_{oD}^{(2)}.$

At stage 1: $q_{1D}^{(1)} = q_{1D}^{(2)}; \qquad q_{1D}^{(2)} = 6.032.$

The resulting Nash profit of each firm $(PV_D)_i$ is

$$(PV_D)_1 = 38.34 \text{ and } (PV_D)_2 = 5.999.$$

Clearly, the profit of firm one under DDIS is more than its corresponding profit under NIS. ∎

In the next section discrete linear Nash game with DDIS is considered The sufficient condition for existence of a solution, on the basis that the space of admissible strategies of both players is linear, will be given.

4.4.2. Discrete LQ Nash Game

Let us consider the discrete linear system

$$x_{k+1} = A_k x_k + B_k u_{1k} + B_{2k} u_{2k}$$

where

$$x_k \in R^{n_k}, \quad u_{ik} \in R^{m_{ik}}$$

and u_{ik} is the control of player i at stage k, for i=1,2.

The objective function of P_i is

$$J_i = \frac{1}{2} x_N' Q_{iN} x_N + \frac{1}{2} \sum_{k=0}^{N-1} (x_k' Q_{ik} x_k + u_{ik}' R_{ik}^{(i)} u_{ik} + u_{jk}' R_{jk}^{(i)} u_{jk})$$

where

$$R_{ik}^{(i)} > 0; \qquad R_{jk}^{(i)}, Q_{ik} \geq 0.$$

If P_2 makes the first move at every stage k, and the information available to both players is memoryless, then $\eta_2(k) = x_k$ and $\eta_1(k) = (x_k, u_{2k})$. Let the strategies of P_1 and P_2 be of the form

$$u_{1k} = L_{1k} x_k + L_{2k} u_{2k}$$

$$u_{2k} = F_k x_k .$$

P_1 will assume $u_{2k} = F_k x_k$, solve a minimization problem, and find the optimal u_{1k} as

$$u_{1k} = -(R_{1k}^{(1)} + B_{1k}' P_{1,k+1} B_{1k})^{-1} (B_{1k}' P_{1,k+1} A_k) x_k$$

where $P_{1,k}$ satisfies

$$P_{1,k} = [\tilde{Q}_{1k} + A_k'P_{1,k+1}A_k - A_k'P_{1,k+1}B_{1k}(R_{1k}^{(1)} + B_{1k}'P_{1,k+1}B_{1k})^{-1}(B_{1k}'P_{1,k+1}A_k)] \quad (4.1)$$

and

$$\tilde{Q}_{1k} = Q_{1k} + F_k'R_{2k}^{(1)}F_k.$$

P_2 will assume $u_{1k} = L_{1k}x_k + L_{2k}u_{2k}$, and solve the following minimization problem: minimize J_2 such that

$$x_{k+1} = (A_k + B_{1k}L_{1k})x_k + (B_{2k} + B_{1k}L_{2k})u_{2k}$$

where

$$J_2 = \frac{1}{2}x_N'Q_{2N}x_N + \frac{1}{2}\sum_{k=0}^{N-1}(x_k'\tilde{Q}_{2k}x_k + u_{2k}'\tilde{R}_{2k}^{(2)}u_{2k} + u_{2k}'M_kx_k + x_k'M_k'u_{2k})$$

and

$$\tilde{Q}_{2k} = Q_{2k} + L_{1k}'R_{1k}^{(2)}L_{1k}$$

$$\tilde{R}_{2k}^{(2)} = R_{2k}^{(2)} + L_{2k}'R_{1k}^{(2)}L_{2k}$$

$$M_k = L_{2k}'R_{1k}^{(2)}L_{1k}.$$

The optimal solution is

$$u_{2k} = -(\tilde{R}_{2k}^{(2)} + \tilde{B}_{2k}'P_{2,k+1}\tilde{B}_{2k})^{-1}(\tilde{B}_{2k}'P_{2,k+1}\tilde{A}_k + M_k')x_k$$

where

$$\tilde{A}_k = A_k + B_{1k}L_{1k}; \qquad \tilde{B}_{2k} = B_{2k} + B_{1k}L_{2k}$$

and $P_{2,k}$ satisfies

$$P_{2,k} = [\tilde{Q}_{2k} + \tilde{A}_k'P_{2,k+1}\tilde{A}_k - (\tilde{A}_k'P_{2,k+1}\tilde{B}_{2k} + M_k)(\tilde{R}_{2k}^{(2)} + \tilde{B}_{2k}'P_{2,k+1}\tilde{B}_{2k})^{-1}(B_{2k}'P_{2,k+1}\tilde{A}_k + M_k')]$$

$$P_{2,N} = Q_{2N}. \quad (4.2)$$

If P_2 chooses $F_k = -(\tilde{R}_{2k}^{(2)} + \tilde{B}_{2k}'P_{2,k+1}\tilde{B}_{2k})^{-1}(\tilde{B}_{2k}'P_{2,k+1}\tilde{A}_k + M_k')$, then sufficient conditions for existence of a Nash equilibrium solution with DDIS can be stated in the following proposition.

<u>Proposition 4.2</u>: If there exists an $(m_{1k} \times n)$ matrix L_{1k} and an $(m_{1k} \times m_{2k})$ matrix L_{2k} such that

$$-L_{1k}+L_{2k}(\tilde{R}^{(2)}_{2k}+B'_{2k}P_{2,k+1}B_{2k})^{-1}(B_{2k}P_{2,k+1}\tilde{A}_k+M'_k) = (R^{(1)}_{1k}+B'_{1k}P_{1,k+1}B_{1k})^{-1}(B'_{1k}\ P_{1,k+1}A_k)$$

where $P_{1,k+1}$, $P_{2,k+1}$ satisfies equations (4.1), (4.2) respectively, then

$$u_{1k} = L_{1k}x_k + L_{2k}u_{2k}$$

$$u_{2k} = -(\tilde{R}^{(2)}_{2k} + \tilde{B}'_{2k}P_{2,k+1}\tilde{B}_{2k})^{-1}(\tilde{B}'_{2k}P_{2,k+1}\tilde{A}_k + M'_k)x_k$$

constitute a Nash equilibrium solution with DDIS.

CHAPTER 5

INFORMATION STRUCTURE,OPTIMAL COORDINATION,AND A GAME MODEL OF DUOPOLY

5.1. Introduction

In this chapter we discuss the idea of using DDIS in inducing the Nash followers in a LF game to behave as members of a team with the leader's objective function as the objective of the team. We introduce a static market model of duopoly where the government interferes in this market. We show that by incorporating the decisions of the Nash duopolists in its strategy, the goverment can always succeed in inducing them to cooperate. In Section 5.2, the incentive problem is formulated as an (n+1)-person LF game with one leader and n Nash followers. The leader's problem is to force the followers to act in such a way that even though each one wants to optimize his objective function, they also optimize the objective function of the leader. In Section 5.3, the objective function of the leader is taken to be a convex combination of the objectives of the followers and a quadratic cost function on the strategy of the leader. The incentive mechanism of the organization is formulated by incorporating the decisions of the followers in the strategy of the leader. By employing such forms of strategies, the leader can force the followers to behave as members of a team, with their composite objective function contained in the objective function of the team. In the first part of Section 5.4, we consider a general static market model of duopoly where the government interfers in the market through adjusting the effective income of the potential buyers of the commodity. We show that by adopting the incentive mechanism described in Section 5.2, the government can enforce the two competing firms to cooperate and achieve the Pareto-optimal solution. In the second part of Section 5.4, we analyze in detail the case of a linear market demand relation and a quadratic cost function. We obtain analytic

solutions for the optimal strategies of the firms and the government, where the optiaml strategy of each firm maximizes in the Nash sense its own payoff function, and also maximizes the government's payoff function, and the optimal strategy of the government maximizes its own objective function (which reflects the welfare of the two firms) and forces cooperation between the two firms. We show that in the limit as the unit cost of applying government control tends to infinity, the enforced cooperative optimal controls and profits tends to the voluntary cooperative ones. We discuss the general properties in terms of marginal cost, price, and the consumers' welfare in the context of this problem.

5.2. Formulation of the Coordination Problem

A multi-person decision problem is a team decision problem if the decision makers (DM's) share a common goal but they have different information structures. A team decision problem arises in an organization in which the objective is to maximize the payoff function of the leader (central manager). If the followers (local managers) share the same objective and they behave in such a way as to maximize the leader's payoff function, then the model of such an organization is a team model.

In general, the followers do not share the same objective with the leader but each has his own objective function which he tries to maximize. In such a case the decision problem for the organization can be formulated as a multi-person game. This multi-person game consists of the leader and the followers. The leader is able to announce his strategy before the other players: hence, he may impose a solution which is favorable to himself. Let the leader choose an overall objective for the organization with which he may coordinate the objective functions of the followers. Now, if the leader can

choose a strategy which induces the followers to behave as if they were members of a team with the le. .er's objective as the goal of the team, then we say that an optimal coordination mechanism exists.

The above discussion can be formalized as

(1) Let $I = \{0,1,\ldots,n\}$ denote the set of DM's where $i=0$ denotes the leader and $i = 1,\ldots,n$ denotes the followers.

(2) Let $U_i = \{u_i\}$, with $i \in I$, denote the decision space of DM_i.

(3) Let $J_i : U \to R$, with $i \in I$, denote the payoff function of DM_i, which is defined on the joint decision space $U = \prod_{i=0}^{n} U_i$.

(4) Let $\gamma_o : H \to U_o$ denote the strategy of the leader, mapping the information space H to the decision space U_o.

(5) The decision problem for the organization can be formalized as an (n+1)-person LF game with DM_o as the leader and $\{DM_i\}_{i=1}^{n}$ as Nash followers. This game is defined as

Definition 5.1: If there exists a mapping $T_i : U_o \to U_i$, for $i = 1,\ldots,n$ such that for any $u_o \in U_o$,

$$J_i(Tu_o;u_o) \geq J_i(Tu_o|u_i;u_o) \qquad \forall u_i \in U_i$$

where $Tu_o|u_i = (T_1u_o,\ldots,T_{i-1}u_o,u_i,T_{i+1}\,u_o,\ldots,T_nu_o)$, and if there exists a $u_{os} \in U_o$ such that

$$J_o(Tu_{os};u_{os}) \geq J_o(Tu_o,u_o)$$

then the strategies $(u_{os},u_{1s},\ldots,u_{ns})$, where $Tu_{os} = (u_{1s},\ldots,u_{ns})$, are called LF strategies with DM_o as the leader and DM_i for $i = 1,\ldots,n$ as Nash followers.

The leader's problem is to design his strategy so as to induce the followers to optimize the leader's objective function while playing optimally with respect to their individual objective functions as well.

5.3. The Coordination Mechanism to Enforce Pareto-Optimality

Let $J_i(u_o, u_1, \ldots, u_n)$ be the payoff function of DM_i for $i = 1, \ldots, n$; and let $J_o = \sum_{i=1}^{n} \alpha_i J_i - \frac{1}{2} c_o u_o^2$ be the payoff function of the leader, where $c_o > 0$, $\alpha_i \geq 0$, and $\sum_{i=1}^{n} \alpha_i = 1$. This objective function reflects the welfare of the entire organization since it comprises a convex combination of the payoff functions of its members plus a quadratic cost on the strategy of the leader. The leader's objective is to see that J_o (which is also the goal of the team) is maximized by all the DM's. To have a well-defined problem let us assume that there exists at least one team strategy $u^t = (u_o^t, \ldots, u_n^t)$ which maximizes J_o over all $u \in U$. The leader will seek a strategy $\gamma_o(\cdot)$ such that if $u_o = \gamma_o(\cdot)$ is substituted in J_i, then DM_i will find a unique solution (u_{iN}) as his optimal strategy in the Nash sense, and this solution will also lead to the maximum value of J_o. This can be mathematically formalized: The leader should choose a strategy $u_o = \gamma_o(\cdot)$ such that when DM_i plays non-cooperatively, he will choose u_{iN} which satisfy

$$J_i(\gamma_o(\cdot), u_{1N}, \ldots, u_{iN}, \ldots, u_{nN}) \geq J_i(\gamma_o(\cdot), u_{1N}, \ldots, u_i, \ldots, u_{nN})$$

and

$$J_o(\gamma_o(\cdot), u_{1N}, \ldots, u_{nN}) = J_o(u_o^t, \ldots, u_n^t).$$

Let the leader adopt a strategy of the form

$$u_o = u_o^t + \sum_{i=1}^{n} A_i (u_i - u_i^t).$$

The leader makes use of the information available to him (the decisions of the followers) by formulating a strategy which is a function of these decisions. By choosing such a representation (the reader is referred to [26] for a game-theoretic interpretation of such a representation), the leader is forcing each of the followers to choose a strategy which is equivalent to u_i^t or else they will be penalized. So if there exists a real sequence $\{A_i\}_{i=1}^n$ such that the optimal response of $DM_i(u_{iN})$ is equivalent to u_i^t, then we say that the coordination mechanism to enforce Pareto-optimality exists. The fact that the leader can enforce cooperation (formation of a team which corresponds to co-operation) among the competing followers has deep economic implications, especially in a market structure.

For cooperation to replace competition among several firms in a market (with no leader to enforce Pareto-optimality), not only must the total maximum profit exceed the combined competitive return, but also none of the participants in the cooperative group must be able to achieve higher profit by means of some feasible strategy while all others stick to the cooperative agreement.

In the case when competition dominates cooperation and the firms play nonefficiently, then the leader (e.g. the government) may interfer to enforce cooperation. If the total maximum profit obtained after taking into account the cost of enforcing the cooperation still exceeds the combined competitive return, then the leader has an incentive to interfere and enforce cooperation, as will be shown in the next section.

5.4. A Market Model of Duopoly and Enforcement of Pareto-Optimality

Let there be two firms which share the production of a commodity (or two undifferentiated commodities) and have access to the same potential buyers. The two firms use the quantity of production as their strategic variable. The demand for this commodity depends on the market price and the consumers' effective income. We assume that the government interferes in the market through the consumers by giving them subsidies or applying income taxes, hence the strategic variable of the government has a direct effect on the income of the consumers.

As a coordinator of the market,[1] the objective function of the government is a convex combination of the payoff functions of the two firms and a quadratic cost function on its control; and its problem is to design a strategy which induces the two firms to behave as members of a team with the objective function of the government as the objective function of the team, while maximizing their payoff functions as well.

5.4.1. General Model

Let q_i stand for the output rate produced by firm i and Q stand for the total marketed quantity. Hence,

$$Q = q_1 + q_2.$$

Let us suppose that the aggregate market demand can be represented by a continuous function as follows

$$p = f_1(Q, I)$$

[1]The government may find it necessary to interfere [illegible] achieve a more efficient economy, especially if the competition [illegible] the participants.

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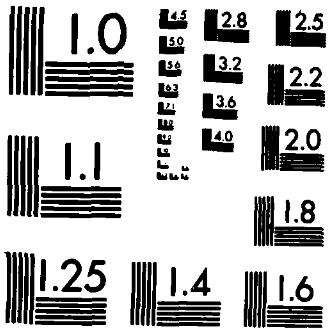

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such that the price (p) is an increasing function of the effective income of the consumers (I), while it is a decreasing function of the demanded quantity (Q).

The effective income of the consumers (I) can be decomposed into two components I_o and i_o, where I_o is the nominal income of the consumers which the government has no control over; and i_o is that part of the consumer's income which the government can directly adjust through taxes or subsidies. Hence i_o can be taken to be the strategic variable of the government. Consequently, equation (5.1) can be reformulated as

$$p = f(Q, i_o).$$

It is reasonable to assume that each firm wants to maximize its net profit J_i, where

$$J_i = pq_i - h_i(q_i).$$

$h_i(q_i)$ is an increasing function in q_i and represents what it costs firm i to produce at a rate q_i.

The objective of the government as the coordinator of the market is to find a joint strategy $(q_{1t}; q_{2t}; i_{ot})$ that maximizes $J_o(q_1, q_2, i_o)$ where

$$J_o(q_1, q_2, i_o) = \alpha_1 J_1 + \alpha_2 J_2 - \frac{1}{2} c_o i_o^2$$

$\alpha_i \geq 0$, $c_o > 0$, and $\alpha_1 + \alpha_2 = 1$.

The optimal joint strategy $(q_{1t}; q_{2t}; i_{ot})$ should satisfy the following

$$\frac{\partial f}{\partial Q} \frac{\partial Q}{\partial q_i} (\alpha_i q_i + \alpha_j q_j) + \alpha_i f - \alpha_i \frac{\partial h_i(q_i)}{\partial q_i} = 0 \qquad \text{for } i,j=1,2 \text{ and } i \neq j \qquad (5.2)$$

and

$$i_o = \frac{\partial f/\partial i_o}{C_o}(\alpha_1 q_1 + \alpha_2 q_2). \tag{5.3}$$

To have a well-defined problem, we are going to assume that there exists at least one joint feasible strategy (by feasible strategy we mean one with non-negative output rates) which achieves the maximum of J_o. Hence, we can see from equation (5.3) that the optimal strategy for the government (i_{ot}) is to give subsidies for the consumers and consequently increase their effective income. The increase of the effective income of the consumers enables them to be willing to buy any fixed quantity at a higher price, which in effect increases the firms' profits. To induce firm i to produce at an output rate which is equal to q_{it} for i=1,2, the government chooses a decision rule (strategy) of the following representation

$$i_o = i_{ot} + A_1(q_1 - q_{1t}) + A_2(q_2 - q_{2t}) \tag{5.4}$$

where A_1 and A_2 are real constants to be chosen by the government to enforce Pareto-optimality. Clearly, if the output rate of firm i equals q_{it} for i=1,2, then $i_o = i_{ot}$ in equation (5.4), that is the strategy of the government will be optimal, and the indirect cooperation between the two firms will also be achieved.

If i_o as given in equation (5.4) is substituted in the payoff function of firm i, then firm i will find its optimal output rate by maximizing its payoff function on the basis that the output rate of the other duopolist is given. The necessary conditions for the above maximization problem can be mathematically described as

$$\frac{\partial J_i}{\partial q_i} = p + q_i\left[\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial q_i} + \frac{\partial f}{\partial i_o}\frac{\partial i_o}{\partial q_i}\right] - \frac{\partial h_i(q_i)}{\partial q_i} = 0 \qquad \text{for } i=1,2 \tag{5.5}$$

where

$$\frac{\partial Q}{\partial q_i} = 1 \quad \text{and} \quad \frac{\partial i_o}{\partial q_i} = A_i.$$

The leader's problem is to choose the real numbers A_1 and A_2, which will cause the optimal response of firm i to be equal to q_{it}. If A_i is chosen to be

$$A_i = \left.\frac{\frac{\partial h_i(q_i)}{\partial q_i} - f(Q,i_o) - q_i\frac{\partial f}{\partial Q}}{q_i\frac{\partial f}{\partial i_o}}\right|_{\substack{q_1=q_{1t}\\ i_o=i_{ot}\\ q_2=q_{2t}}} \tag{5.6}$$

then the optimal output rate in the Nash sense of firm i will equal q_{it}. We conclude this section by the following proposition, which we have just proved.

Proposition 5.1: In a market duopoly, by declaring a strategy of the form given in equation (5.4), with A_i satisfying equation (5.6), the government can force the two firms to choose a Pareto-optimal output rate, while maximizing their corresponding profit as well.

5.4.2. A Linear Model Case

In this section, we will determine analytic solutions for the coordination problem presented in the last section. This will aid in understanding the economic implication of the coordination problem.

Let the market demand relation satisfy the following linear equation (for a discussion of the linear demand relation, the reader is referred to [36]

$$Q = d_o + d_1 p + d_2 i_o$$

where d_o and d_2 are positive constants, while d_1 is a negative constant. Let the cost function $h_i(q_i)$ be of the form

$$h_i(q_i) = \frac{1}{2} C_i q_i^2 + K_i$$

where C_i and K_i are positive constants. For the linear model case, we have the following corollary.

Corollary 1: In a market duopoly with a linear demand relation and a quadratic cost function, the government can force cooperation between the two firms by choosing A_i to be

$$A_i = \frac{q_{it}(2-c_i d_1) + q_{jt} - d_o + d_2 i_{ot}}{d_2 q_{it}} \quad \text{for } i,j=1,2 \text{ and } i \neq j$$

where

$$q_{it} = \frac{b_{ij}\bar{a}_j^{-1} + S_i}{\bar{a}_i - b_{ij}^2 \bar{a}_j^{-1}} \quad ; \quad i_{ot} = \frac{-d_2}{d_1 c_o}(\alpha_1 q_{1t} + \alpha_2 q_{2t})$$

and

$$b_{ij} = -1 - \frac{\alpha_i \alpha_j d_2^2}{d_1 c_o} \quad ; \quad s_i = d_o \alpha_i$$

$$\overline{a_i} = 2\alpha_i - \alpha_i d_1 c_i + \frac{\alpha_i^2 d_2^2}{d_1 c_o} .$$

Proof: The proof is straightforward and follows the same lines given in the discussion of the general case. ∎

Instead of absorbing the total cost of cooperative enforcement, the government may charge the firms the cost of policing the cooperation. Let the cost charged to firm i be $\frac{1}{2}\bar{c}_i i_o^2$, where $\alpha_1 \bar{c}_1 + \alpha_2 \bar{c}_2 = c_o$. The objective function of the government (J_o) will be

$J_o = \alpha_1 J_1 + \alpha_2 J_2$, where the profit of firm i (J_i) is

$$J_i = pq_i - \frac{1}{2} c_i q_i^2 - \frac{1}{2} \bar{c}_i i_o^2 .$$

It can be shown that the government can force cooperation between the two firms by choosing A_i of a slightly different form,

$$A_i = \frac{q_{it}(2-c_i d_1) + q_{jt} - d_o + d_2 i_{ot}}{d_2 q_{it} + 2\bar{c}_i i_{ot}}\,, \tag{5.7}$$

where q_{it}, q_{jt}, and i_{ot} are as previously described.

In the remainder of the chapter we consider the important special case when the objective function of the government (J_o) is the total profit of the two firms and a quadratic cost on the strategy of the government, i.e., $J_o = J_1 + J_2 - \frac{1}{2} c_o i_o^2$. By the same techniques as above, we find that

$$q_{it} = \frac{-d_1 d_o c_j c_o}{c_o[d_1^2 c_1 c_2 - 2(c_1+c_2)d_1] - (c_1+c_2)d_2^2} \tag{5.8}$$

$$i_{ot} = \frac{d_2 d_o (c_1+c_2)}{c_o[d_1^2 c_1 c_2 - 2(c_1+c_2)d_1] - (c_1+c_2)d_2^2} \tag{5.9}$$

and

$$A_i = \frac{c_o(c_i d_1) + 2d_2^2(c_i+c_j)}{-d_1 d_2 c_J c_o}\,. \tag{5.10}$$

q_{it}, i_{ot}, and A_i are sketched as a function of the parameter c_o in Fig. 5.1, Fig. 5.2, and Fig. 5.3, respectively, where c_o denotes the cost of applying one unit of government control.

From Fig. 5.1 it can be seen that the optimal output rate of the firms are feasible (the q_i's are non-negative) if and only if

$$c_o > \frac{(c_1+c_2)d_2^2}{d_1^2 c_1 c_2 - 2(c_1+c_2)d_1}\,.$$

Hence, the optimal feasible strategy of the government amounts to increasing the effective income of the buyers as shown in Fig. 5.2.

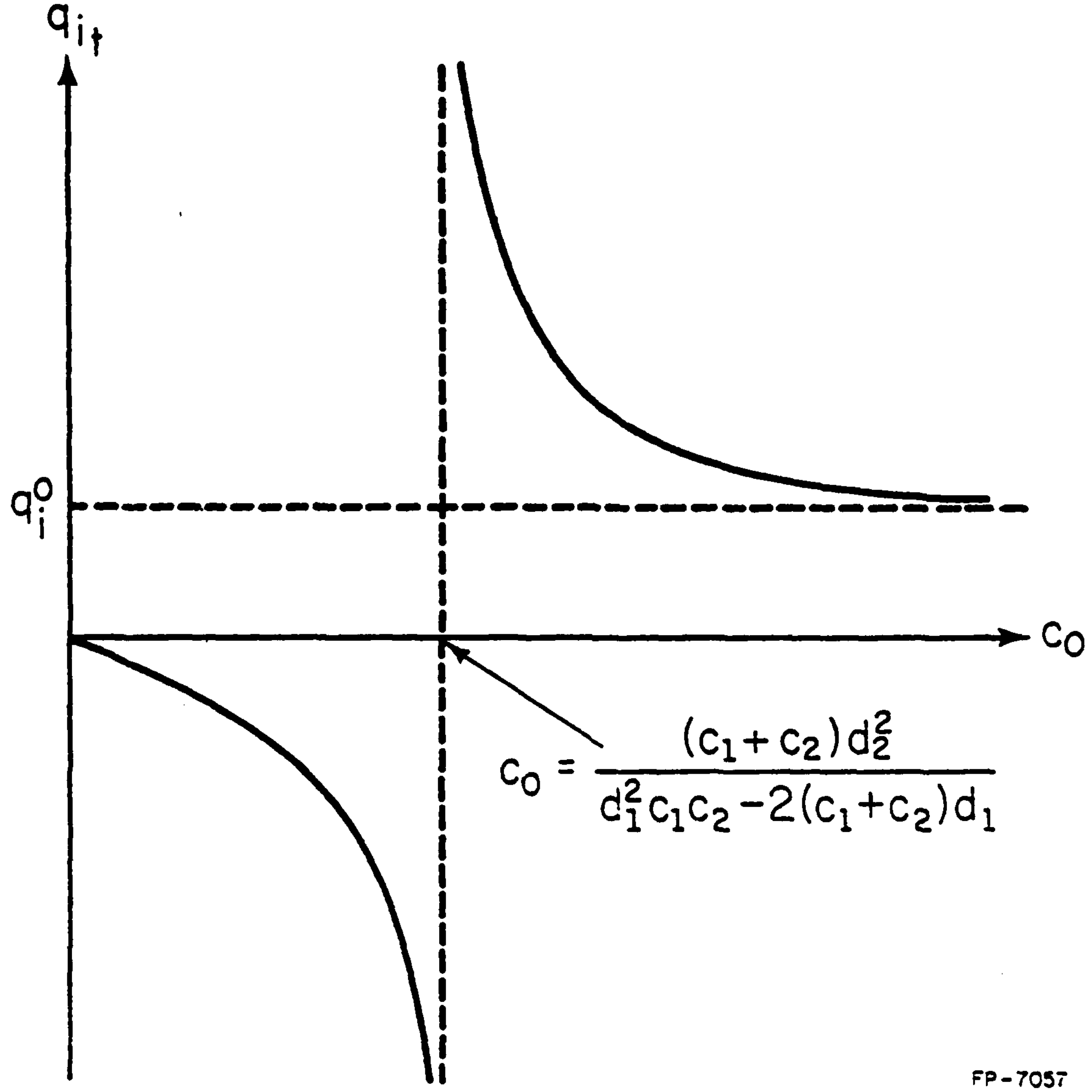

Figure 5.1. The optimal output rate of firm i vs. the cost of one unit of government control.

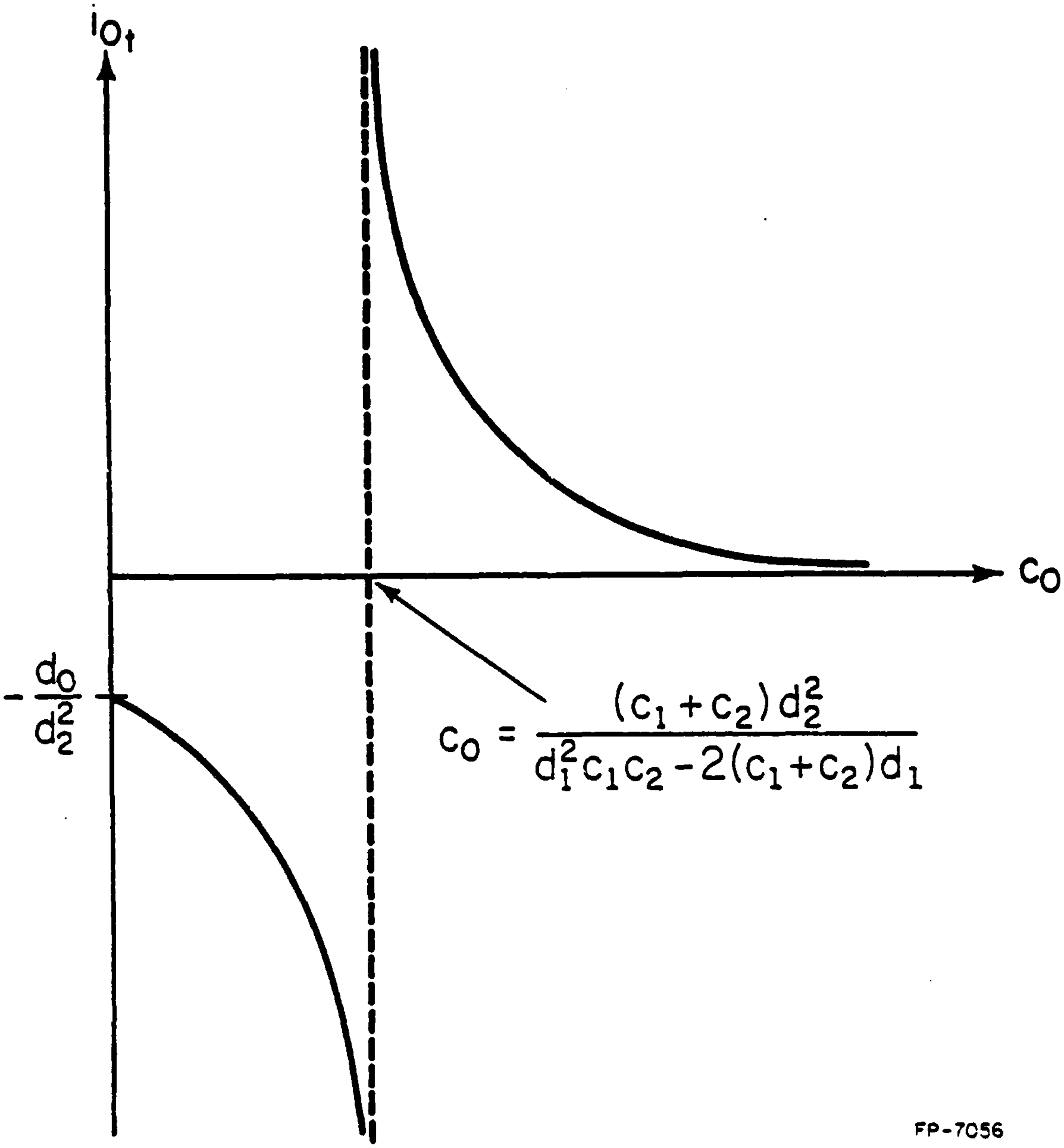

Figure 5.2. The optimal government control vs. its one unit cost.

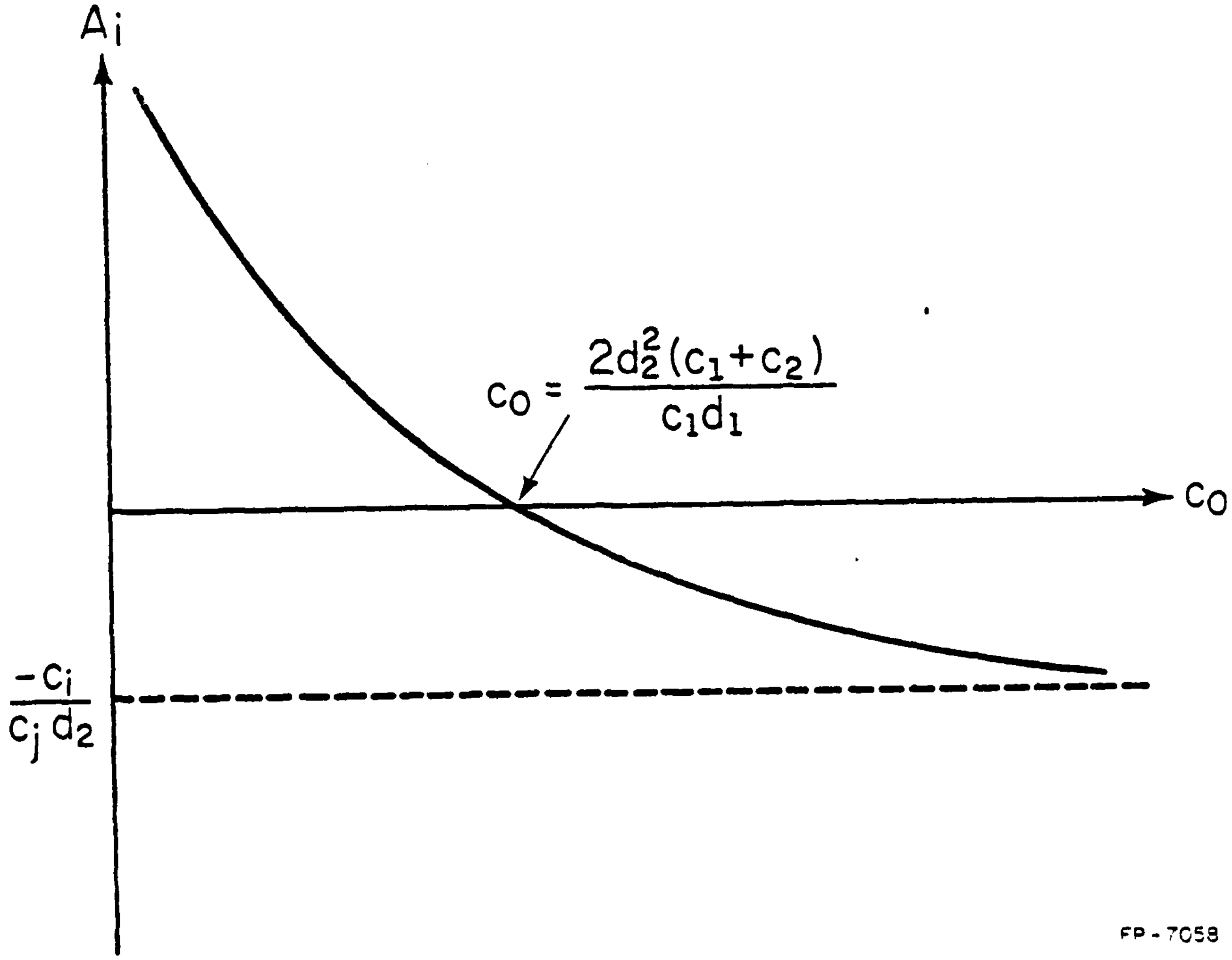

Figure 5.3. The optimal constant A_i vs. the cost of one unit of government control.

5.4.2.1. Forced Cooperation vs Voluntary Cooperation

If the two firms agree to cooperate with J_1+J_2 as their objective function (they agree to form a cartel, where the members retain their separate identities and separate control over their policies), then the optimal output of the two members of the group can be found by solving a straightforward maximization problem of the total profit of the two firms on the basis of the demand conditions. This optimal output rate is

$$q_i^o = \frac{-c_j d_1 d_o}{-2c_1d_1 - 2c_2d_1 + d_1^2 c_1 c_2} \quad \text{for } i,j=1,2 \text{ and } i \neq j$$

and J_o^o is the optimal total profit. The trouble with this solution is that it may not be practically attained, since any firm can depart from this solution without being penalized for increasing its profits. For example, if firm 1 supposes that firm 2 is going to stick to the cooperative solution $q_2^o = \dfrac{-c_1d_1d_o}{-2c_1d_1-2c_2d_1+d_1^2c_1c_2}$, firm 1 chooses its optimal output rate as the one which satisfies its rational reaction relation $q_1 = \dfrac{d_o}{2-c_1d_1} - \dfrac{1}{2-c_1d_1} q_2$. Then we have $q_1 = \dfrac{d_o(-c_1d_1-2c_2d_1+d_1^2c_1c_2)}{(2-c_1d_1)(-2c_1d_1-2c_2d_1+d_1^2c_1c_2)}$. Hence, by departing from the cooperative solution, firm 1 managed to increase its profits. Of course, the same argument applies to firm 2. As a result no firm may be willing to join the coalition, knowing in advance that it and all others may depart from the cooperative agreement.

Earlier in this chapter, we showed that the government can force cooperation between the two firms if it announces a strategy of the form given in equation (5.4), where A_i satisfies equation (5.10). The optimal output

rate of the two firms and the strategic variable of the government depend on c_o. If c_o is large enough, then the total profit (J_o) of the two firms under forced cooperative conditions, will become arbitrarily close to their total profit (J_o^o) under the voluntary coalition conditions, since it can be seen that as $c_o \to \infty$, $q_{it} \to q_i^o$, $i_{ot} \to 0$, and $J_o \to J_o^o$.

For large enough c_o there is a seeming paradox, since, although in both situations of voluntary and enforced cooperation, the government contributes nothing or almost nothing to the market, and each firm has the same output rate, both firms have an incentive to depart from the Pareto-optimal solution under voluntary cooperative conditions; whereas no firm would change its solution under enforced cooperative conditions, since it can only lose by departing from the equilibrium solution. The answer to this paradox is that the strategy of the government has two different representations for the same value in these two cases ($i_o = 0$ for the voluntary case, while $i_o = A_1(q_1 - q_{1t}) + A_2(q_2 - q_{2t})$ for the enforced cooperation). The representation of the strategy in the enforced Pareto-optimal case has a threatening power, which directs each firm to behave in a certain way or else it will incur additional costs.

In summary, we can say that by being in a coordinating position in the market, the government forces the two firms to cooperate, thus obtaining the maximal monopolist joint profit, but with arbitrarily small cost.

The following numerical example illustrates the basic ideas presented in this section.

5.4.2.2. Numerical Example

Let the market demand relation be

$$q_1 + q_2 = 10 - 2p + i_o.$$

Let us assume that the two firms have identical cost functions with

$$c_1 = c_2 = 1.$$

Substituting the values of d_o, d_1, d_2, c_1, c_2 in equations (5.7), (5.8), (5.9), we find that

The optimal output rate of each firm $(q_{it}) = \dfrac{20c_o}{12c_o-2}$.

The value of the control of the government $(i_{ot}) = \dfrac{10}{6c_o-1}$.

The strategy of the government (i_o) is

$$i_o = \left(\frac{10}{6c_o-1}\right) + A_i[(q_1-q_{1t})+(q_2-q_{2t})]$$

where $$A_i = \frac{-c_o+2}{c_o}.$$

For $c_o = 10$, we find that

$$q_{it} = 1.695; \qquad i_{ot} = 0.1695$$

$$A_1 = A_2 = -0.8$$

$$p = 3.39$$

$$J_1+J_2 = 8.6190; \qquad \frac{1}{2}c_o i^2 = 0.1436.$$

Hence,

$$J_o = J_1+J_2-\frac{1}{2}c_o i^2 = 8.475.$$

For $c_o = 1000$, we find that

$$q_{it} = 1.667; \qquad i_{ot} = 0.0017$$

$$A_i = -0.998, \quad p = 3.333; \quad \frac{1}{2}c_o i_o^2 = 0.0014$$

$$J_1+J_2 = 8.333 \quad \text{and} \quad J_o = 8.332.$$

If the two firms agreed to cooperate (without the interference of the government), then the Pareto-optimal solution would be

$$q_1^o = q_2^o = 1.667$$

$$i_o^o = 0$$

and

$$J_o = J_1 + J_2 = 8.333.$$

It can be seen from the above that for $c_o = 10$, the total profits after taking into account the cost of policing the cooperation is bigger than the total profits under voluntary conditions. In such cases the government may charge the two firms the cost of its control, for example when $\bar{c}_1 = \bar{c}_2 = 5$ with $\bar{c}_1 + \bar{c}_2 = c_o$, the government, by choosing $A_i = -0.3998$ (see equation (5.7)), can enforce cooperation between the two firms, charge them the cost of its control, and still make the total profit of the two firms exceed the voluntary monopolistic total profit. By increasing the consumers' demand for the marketed commodity, the government altered the market structure and consequently, increased the total profit of the two firms compared to their profit under volunary cooperative conditions.

For large enough values of c_o ($c_o = 1000$ is large enough for our example), the values of the optimal strategies and profits of the firms and government under forced cooperation conditions are arbitrarily close to the corresponding values under voluntary cooperation conditions.

The optimal solution under enforced cooperation conditions is a LF equilibrium solution with the government as a leader and the two firms as Nash followers. Hence, there is no incentive for any participant to leave this solution.

Before concluding this section, several remarks should be mentioned.

R5.1 If we compare the output rate and the price in the market under forced cooperation and voluntary cooperation, we find that

$$Q^o = q_1^o + q_2^o = \frac{-d_1 d_o(c_1+c_2)}{-2c_1d_1-2c_2d_1+d_1^2c_1c_2}$$

and

$$p^o = \frac{d_o(c_1+c_2)-d_1d_oc_1c_2}{-2c_1d_1-2c_2d_1+d_1^2c_1c_2}$$

while

$$Q_t = q_{1t}+q_{2t} = \frac{-d_1d_o(c_1+c_2)c_o}{c_o[-2c_1d_1-2c_2d_1+d_1^2c_1c_2]-(c_1+c_2)d_2^2}$$

and

$$p_t = \frac{c_o[d_o(c_1+c_2)-d_1d_oc_1c_2]}{c_o[-2c_1d_1-2c_2d_1+d_1^2c_1c_2]-(c_1+c_2)d_2^2} .$$

For all feasible values of $c_o\left(c_o > \left(\frac{(c_1+c_2)d_2^2}{d_1^2c_1c_2-2(c_1+c_2)d_1}\right)\right)$ the output rate and the price under forced cooperation are larger than the corresponding output rate and price under voluntary cooperation. The welfare of the consumers (W) is appropriately defined as the consumer's surplus which can be mathematically formalized as $W(Q_N) = \int_0^{Q_N}(p(s)-P_N)ds$, where P_N and Q_N are the equilibrium values of price and quantity, respectively. It can be verified that W is worse off under voluntary cooperation, hence, the government increases the social welfare of the consumers compared to that under voluntary cooperative conditions. But, in the limit as $c_o \to \infty$, the price, the output rate, and the social welfare of the consumers will be equal under both conditions.

R5.2 The price elasticity of demand (η) at both the voluntary cooperative point and for all values of c_o at the enforced cooperative point is identical and it equals

$$\eta = (d_o(c_1+c_2)-d_1d_oc_1c_2)/(d_o(c_1+c_2)).$$

It can be seen that the market is always price elastic under both conditions.

R5.3: Under voluntary cooperative conditions the output rate (q_i^o) satisfies

$$p^o - c_iq_i = \frac{-q_i^o}{d_1}\left(1+\frac{c_i}{c_j}\right) \qquad \text{for } i=1,2.$$

Under forced cooperative conditions, the output rate (q_{it}) satisfies

$$p_t - c_iq_{it} = \frac{-1}{d_1}\,q_{it}\left(1+\frac{c_oc_id_1+2d_2^2(c_1+c_2)}{d_1^2c_jc_o}\right) \qquad \text{for } i=1,2.$$

From the above equations, it can be shown that under voluntary cooperative conditions, the price (p) is higher than the marginal cost (MC) for both firms; whereas under enforced cooperation conditions, the price (p) is higher than the marginal cost (MC) for both firms if and only if $c_o > \frac{2d_2^2}{-d_1}$. (The feasibility conditions is also satisfied since $\frac{2d_2^2}{-d_1} > \frac{(c_1+c_2)d_2^2}{d_1^2c_1c_2-2(c_1+c_2)d_1}$.) For each firm the difference between p and MC is smaller for the enforced solution than for the voluntary one; thus the former solution is closer to the group rationality solution (to satisfy group rationality, which includes the buyers, the price must equal marginal cost) than the latter.

R5.4: It can be noticed from the previous example that depending on c_o, the total profits of the two firms under enforced cooperation conditions may be larger, equal, or smaller than the corresponding profits under voluntary conditions. The procedure to find the region of c_o for which that is true, is straightforward but lengthy. It will not be pursued here.

R5.5: Our detailed analysis was constrained to the case where the objective function of the government (J_o) is the total profit of the two firms and a quadratic cost on the strategy of the government, similar results are expected when the objective function of the government (J_o) is the convex combination of the profit functions of the two firms and a quadratic cost on the control of the government, i.e., $J_o = \alpha_1 J_1 + \alpha_2 J_2 - \frac{1}{2} c_o i_o^2$.

CHAPTER 6

ON SOME STOCHASTIC STATIC TEAM LEADER-FOLLOWER PROBLEMS

6.1. Introduction

In this chapter we deal with two static, stochastic leader-follower team problems, where each player has a quadratic cost function and part of his information is a linear function of Gaussian random variables. In Section 6.2, we consider the problem of a 3-person stochastic optimal coordination, where the coordinator desires to induce the two noncooperative (in Nash sense) players to minimize his cost function, even though each player minimizes his own cost function. The cost function of the coordinator is a convex combination of the noncooperative players' cost functions. The information structure of the game is nested and dynamic, whereby the coordinator not only knows whatever the other players know, but also detect exactly their decision variables. We show that by incorporating the decisions of the other players, the coordinator can, under a certain condition, successfully enforce the team solution with his objective as the goal of the team.

In Section 6.3, we consider a two-person leader-follower game, in which the leader does not completely detect the decision variable of the follower. To achieve the best possible outcome of the leader, we define a new modified team problem after taking into account the optimal response of the undetected action of the follower. We find that the leader can, under a ceratin condition, achieve this new tight lower bound. Finally, we give a numerical example to illustrate our procedure.

6.2. Stochastic Optimal Coordination

Let x be a Gaussian random vector over a probability space $(\Omega,\mathcal{F},P)$, $x:\Omega\rightarrow R^n$ with zero mean and Σ covariance. Let u_i denote the decision variable of decision maker $(DM)_i$ which takes values in R^{m_i} for $i=0,1,2$. The objective functional of $(DM)_i$ for $i=1,2$ is defined by

$$G_i(x,u_o,u_1,u_2) = \frac{1}{2}\, u_i'D_iu_i + u_i'Q_ix + u_i'R_iu_j + \frac{1}{2}\, u_o'S_iu_o + u_o'L_iu_i + u_o'T_ix.$$

The objective functional of DM_o (the coordinator) is a convex combination of the objective functionals of the other two DM's, i.e.

$$G_o(x,u_o,u_1,u_2) = \alpha_1G_1(x,u_o,u_1,u_2) + \alpha_2G_2(x,u_o,u_1,u_2)$$

where $1\geq\alpha_i\geq 0$, and $\alpha_1+\alpha_2=1$. To describe the information structure of the game, let y_1 and y_2 be two random vectors defined as

$$y_i = H_ix + w_i \qquad \text{for } i=1,2$$

where H_i is $r_i\times n$ real constant matrix, and $w_i\sim N(0,\Lambda_i)$.

Let η_i denote the information available to DM_i. We consider in this section the following information structure

$$\eta_o = y_1,y_2,u_1,u_2$$

$$\eta_1 = y_1$$

$$\eta_2 = y_2.$$

We denote an admissible strategy of DM_i by γ_i which is a Borel measurable mapping from the information space into the decision space, and we also denote the space of admissible strategies of DM_i by Γ_i.

Let us define the expected value of $G_i(x,\gamma_o(\eta_o),\gamma_1(\eta_1),\gamma_2(\eta_2))$ with respect to the random variable x by

$$J_i(\gamma_o,\gamma_1,\gamma_2),\ \text{i.e.}\quad J_i(\gamma_o,\gamma_1,\gamma_2) = E\{G_i(x,\gamma_o(\eta_o),\gamma_1(\eta_1),\gamma_2(\eta_2))\}$$

The coordinator's goal is the achievement of the global minimum of J_o, and his role is to adopt a strategy, $\tilde{\gamma}_o(\eta_o)$, which induces DM_1 and DM_2 to play with him as a team with J_o as the objective function, even though there is no explicit cooperation between them. If $\tilde{\gamma}_o(\eta_o)$ is substituted in J_i, and J_i is minimized over $\gamma_i(\eta_i)$, in the Nash sense, and if the resulting solution leads to the minimization of J_o, then the optimal coordination problem is solved.

By Nash solution we mean that if $\gamma_o(\eta_o)$ is a given strategy, then

$$J_1(\gamma_o,\gamma_1^o,\gamma_2^o) \le J_1(\gamma_o,\gamma_1,\gamma_2^o)$$
$$J_2(\gamma_o,\gamma_1^o,\gamma_2^o) \le J_2(\gamma_o,\gamma_1^o,\gamma_2).$$

6.2.1. Derivation of the Optimal Equilibrium Strategies

We start by finding the optimal team solution. Let

$$\min_{\gamma_o\in\Gamma_o}\ \min_{\gamma_1\in\Gamma_1}\ \min_{\gamma_2\in\Gamma_2} J_o(\gamma_o,\gamma_1,\gamma_2) = J_o(\gamma_o^t,\gamma_1^t,\gamma_2^t).$$

Lemma 6.1: If

$$\begin{vmatrix} \tilde{S}_o & \alpha_1 L_1 & \alpha_2 L_2 \\ \alpha_1 L_1' & \alpha_1 D_1 & \alpha_1 R_1 + \alpha_2 R_2' \\ \alpha_2 L_2' & \alpha_1 R_1' + \alpha_2 R_2 & \alpha_2 D_2 \end{vmatrix} > 0$$

then the globally optimal team solution exists and is given by

$$\gamma_o^t(\eta_o) = -\tilde{S}_o^{-1}[\alpha_1 L_1 u_1 + \alpha_2 L_2 u_2 + \tilde{T}_o E(x|y_1,y_2)]$$

where $\tilde{T}_o = \alpha_1 T_1 + \alpha_2 T_2$ and $\tilde{S}_o = \alpha_1 S_1 + \alpha_2 S_2$

$$\gamma_i^t(\eta_i) = K_i y_i \quad \text{for } i = 1,2$$

where K_i satisfies the following matrix equation

$$K_i = -(\alpha_i D_i - \alpha_i^2 L_i' \tilde{S}_o^{-1} L_i)^{-1}[(\alpha_i Q_i - \alpha_i L_i' \tilde{S}_o^{-1} \tilde{T}_o) H_i' \Sigma (H_i \Sigma H_i' + \Lambda_i)^{-1}$$

$$+ (\alpha_i R_i + \alpha_j R_j' - \alpha_i \alpha_j L_i' \tilde{S}_o^{-1} L_j) K_j H_j \Sigma H_i' (H_i \Sigma H_i' + \Lambda_i)^{-1}].$$

Proof: The proof is straightforward and can be obtained by using standard stochastic control techniques under the nested information structure consideration [48,43]

The optimal team solution can be expressed as

$$\gamma_o^t(y_1,y_2,u_1,u_2) = h_{o1} y_1 + h_{02} y_2 + h_{o3} u_1 + h_{o4} u_2.$$

$$\gamma_i^t = K_i y_i \qquad \text{for } i = 1,2.$$

The coordinator's problem is to adopt a certain form of strategy, in which he incorporates the decisions of DM_1 and DM_2, and by which he can force the two DM's to cooperate while each DM is minimizing his cost function as well. Let us restrict our investigations to the class of strategies which are linear in u_1 and u_2

$$\gamma_o(\eta_o) = \gamma_o^t(\eta_o) + A_1(u_1 - \gamma_1^t) + A_2(u_2 - \gamma_2^t).$$

In general we can take A_1 and A_2 as matrix functions which are measurable with respect to the sigma fields generated by η_1 and η_2 respectively. But, for simplicity we will consider only constant matrices. It is clear that if

the matrices A_1 and A_2 are chosen in such a way that the optimal response of DM_1 and DM_2, u_1 and u_2 equal their corresponding team decisions, then we say that the stochastic optimal coordination problem is solved.

To find the rational reaction of DM_i to the announced strategy of the coordinator, we substitute $\gamma_o(\eta_o)$ as given above in $J_i(\gamma_o,u_1,u_2)$. Since $J_i(\gamma_o(\eta_o),u_1,u_2)$ is quadratic and if strict convexity of J_i on Γ_i is assumed, then the minimum of J_i with respect to u_i can be obtained by person-to-person optimization and by taking the gradient of J_i with respect to u_i and equating it to zero. If this procedure is carried out it follows that

$$\begin{aligned} E[\{D_iu_i+Q_ix+R_iu_j+(h_{oi}+A_i)'S_i[(h_{oi}+A_i)(u_i-K_iy_i)+(h_{oj}+A_j)(u_j-K_jy_j) \\ +h_{oi+2}y_i+h_{oj+2}y_j] + L_i'(A_i(u_i-K_iy_i) + A_j(u_j-K_jy_j) + h_{oi}u_i+h_{oj}u_j \\ +h_{03}y_1+h_{o4}y_2) + (A_i+h_{oi})'L_iu_i + (A_i+h_{oi})'T_ix\}|y_i] = 0. \end{aligned}$$

After some straightforward manipulation we find that

$$\gamma_i^*(y_i) = -M_i^{-1}V_{ij}E(x|y_i) - N_{ij} - O_{ij}E(\gamma_j^*(y_j)|y_i) \tag{6.1}$$

where

$$\begin{aligned} M_i &= D_i + (h_{oi}+A_i)'S_i(h_{oi}+A_i) + L_i(A_i+h_{oi}) + (A_i+h_{oi})'L_i \\ V_{ij} &= Q_i + (h_{oi}+A_i)'S_ih_{o,2+j}H_j + L_i'h_{2+j}H_j-(h_{oi}+A_i)'S_i(h_{oj}+A_j)K_jH_j \\ &\qquad -L_i'A_jK_jH_j + (A_i+h_{oi})'T_i \\ N_{ij} &= (L_i'h_{2+i}-(h_{oi}+A_i)'S_i(h_{oi}+A_i)K_i-L_i'A_iK_i + (h_{oi}+A_i)'S_ih_{o2+i})y_i \\ O_{ij} &= [R_i + (h_{oi}+A_i)'S_i(h_{oj}+A_j) + L_i'(A_j+h_{oj})]. \end{aligned}$$

That M_i is a nonsingular matrix and follows from the strict convexity assumption of J_i on Γ_i.

The following proposition gives the conditions of existence and uniqueness of the solution of (6.1).

Proposition 6.1: If either

$$\bar{\lambda}(M_i^{-1}O_{ij}M_j^{-1}O_{ji}) < 1$$

or

$$\bar{\lambda}(M_j^{-1}O_{ji}M_i^{-1}O_{ij}) < 1$$

then, there exists a unique linear Nash solution (γ_1^*,γ_2^*) which satisfies equation (6.1), for each pair (A_1,A_2), where $\bar{\lambda}(A)$ is the maximum eigenvalue of any bounded real symmetric matrix (A'A).

Proof: The proof is equivalent to the one given in [49].

Let $\gamma_i^*(y_i) = K_i y_i$ in (6.1), then we have

$$K_i = -M_i^{-1}V_{ij}\Sigma_i - M_i^{-1}\bar{N}_{ij} - M_i^{-1}O_{ij}K_j\Sigma_i \tag{6.2}$$

where

$$\Sigma_i = \Sigma H_i'(H_i\Sigma H_i' + \Lambda_i)^{-1}.$$

Equation (6.2) can be rearranged and written as follows

$$A_i'[L_iK_i + T_i\Sigma_i + S_i(h_{o,2+i}+h_{0,2+j}H_j\Sigma_i)] + D_iK_i + Q_i\Sigma_i + h_{oi}'[L_iK_i + T_i\Sigma_i$$
$$+ S_i(h_{o2+i}+h_{o2+j}H_j\Sigma_i)] + L_i'(h_{o2+i}+h_{o2+j}H_j\Sigma_i)$$
$$+ R_iK_jH_j\Sigma_i + L_i'h_{oj}K_jH_j\Sigma_i = 0$$

or equivalently

$$A_i'\bar{B}_o^i = \bar{B}_1^i.$$

Condition 6.1: There exists at least one matrix A_i for $i = 1,2$ which satisfies

$$A_i'\bar{B}_o^i = \bar{B}_1^i.$$

The dimensions of A_i', $\bar{B}_o^i$, and $\bar{B}_1^i$ are $m_i \times m_o$, $m_o \times r_i$, $m_i \times r_i$, respectively, and we have $m_i \times m_o$ variables to choose which satisfy $m_i \times r_i$ equations. Condition 6.1 is generally guaranteed if $m_o \geq \max(r_1, r_2)$.

From before, we can easily see the proof of the following theorem.

Theorem 6.1: If Condition 6.1 is satisfied and if $\gamma_o(\eta_o) \in \Gamma_o$ is picked as

$$\gamma_o(\eta_o) = \gamma_o^t + A_1(u_1 - \gamma_1^t) - A_2(u_2 - \gamma_2^t)$$

then a solution to the optimal coordination problem exists

6.3. Linear Quadratic Leader-Follower Games with Partial Decision-Dependent Information Structure

As in the previous section, x-n-dimensional random vector and $x \sim N(0, \Sigma)$. u_i is the decision variable of player i (P_i), and it takes values in R^{m_i} for $i = 1,2$. Let P_1 be the leader while P_2 is the follower. The objective of the leader is defined as

$$L_1(x, u_1, u_2) = \frac{1}{2} u_1' R_{11} u_1 + u_1' R_{12} u_2 + u_1' R_{13} x + \frac{1}{2} u_2' R_{14} u_2 + u_2' R_{15} x,$$

while the objective function of the follower is

$$L_2(x, u_1, u_2) = u_2' R_{21} x + \frac{1}{2} u_2' R_{22} u_2 + u_1' R_{23} x + u_1' R_{24} u_2,$$

where

$$R_{11}, R_{22} > 0.$$

Let y_1 and y_2 be two random variables defined by

$$y_i = H_i x + w_i \qquad \text{for } i = 1,2$$

where H_i is $r_i \times n$ real constant matrix, and $w_i \sim N(0, \Lambda_i)$. The leader detects the action of the follower through observing y_o, which can be taken without

loss of generality as

$$y_o = [I \quad 0]u_2$$

where I is r×r-identity matrix with $r < m_2$. Let η_i denote the information available to P_i, and we are going to assume the following nested information structure

$$\eta_1 : y_o, y_1, y_2$$

$$\eta_2 : y_2.$$

As before, γ_i is a Borel measurable mapping from the information space to the decision space of P_i. The space of all such mappings is the strategy space of $P_i(\Gamma_i)$.

One natural lower bound for J_1 is the infimum of J_1 over the product set $\Gamma_1 \times \Gamma_2$ (J_1^t). Let us assume that this infimum is actually achieved, so the leader wants to adopt a certain strategy by which he can force the follower to play in such a way so as to globally minimize J_1, even though the follower intends to minimize his cost function. To investigate the ability of the leader to enforce the team solution, let us make the following assumptions.

Assumption A: The control value of the follower can be exactly and completely detected by the leader through his information.

Assumption B: There exists a strategy for the leader by which he can increase the follower's cost, if the latter does not abide by the team solution.

Clearly, if the above two assumptions are satisfied then the leader can force his team solution (see [47] for details).

In this section we are dealing with the case when assumption A is not satisfied, thus J_1^t is not the realizable tight lower bound for J_1. This problem can be dealt with by defining a new modified team problem for the leader which takes into account the rational reaction of the follower on the undetectable action space.

The procedure to solve the problem when the complete detectability condition is not satisfied is as follows.

1. Minimize $J_2(\gamma_{22},\gamma_{21},\gamma_1)$ with respect to $\gamma_{22}\in\Gamma_{22}$, where

$$\gamma_2 = \begin{bmatrix}\gamma_{21}\\ \gamma_{22}\end{bmatrix}, \qquad \gamma_{22}\in R^{m_2-r} \text{ and } \Gamma_2=\Gamma_{21}\cup\Gamma_{22}$$

Since J_2 is quadratic in γ_{22}, under strict convexity assumption, we obtain a unique optimal strategy γ_{22}^* which is given as the unique map f, where

$$\gamma_{22}^* = f(\gamma_{21},E(\gamma_1|y_2),\ E(x|y_2)).$$

2. Substitute for γ_{22}^* in J_1 to obtain a new cost functional $\tilde{J}_1(\gamma_{21},\gamma_1,E(\gamma_1|y_2),E(x|y_2))$. The infimum of $\tilde{J}_1$ over the product set $\Gamma_1\times\Gamma_{21}$ is the new lower tight bound (J_1^*). If the infimum is actually achieved, then the leader can force this modified team solution as will be shown in the following.

6.3.1. Determination of the New Tight Lower Bound of $\tilde{J}_1$

1. Minimizing $J_2(\gamma_{22},\gamma_{21},\gamma_1)$ with respect to γ_{22}, we find that

$$\gamma_{22}^*(y_2) = -R_{22}^{(3)^{-1}}[R_{22}^{(2)'}\gamma_{21}+R_{21}^{(2)}E(x|y_2)+R_{24}^{(2)'}E(\gamma_1(y_1,y_2,y_3)|y_2)] \tag{6.3}$$

where

$$R_{21} = \begin{bmatrix} R_{21}^{(1)} \\ \\ R_{22}^{(2)} \end{bmatrix} \quad ; \quad R_{22} = \begin{bmatrix} R_{22}^{(1)} & R_{22}^{(2)} \\ \\ R_{22}^{(2)'} & R_{22}^{(3)} \end{bmatrix}$$

$$R_{24} = [R_{24}^{(1)} \quad R_{24}^{(2)}].$$

Since J_2 is quadratic in γ_{22}, and under the assumption that $R_{22}^{(3)}$ is positive-definite, the minimization is obtained by person-to-person optimization.

2. Substituting the value of $\gamma_{22}^{*}(y_2)$, which is obtained above, in J_1, we get

$$\tilde{J}_1(x,\gamma_1,\gamma_{21},E(\gamma_1|y_2),E(x|y_2)) = E\{\tfrac{1}{2}\gamma_1'R_{11}\gamma_1 + \gamma_1'R_{12}^{(1)}\gamma_{21}$$
$$-\gamma_1'R_{12}^{(2)}R_{22}^{(3)^{-1}}(R_{22}^{(2)'}\gamma_{21}+R_{21}^{(2)}E(x|y_2) + R_{24}^{(2)'}E(\gamma_1|y_2)) + \gamma_1'R_{13}x$$
$$+\tfrac{1}{2}\gamma_{21}'R_{14}^{(1)}\gamma_{21}-\gamma_{21}'R_{14}^{(2)}R_{22}^{(3)^{-1}}(R_{22}^{'(2)}\gamma_{21}+R_{21}^{(2)}E(x|y_2) + R_{24}^{(2)'}E(\gamma_1|y_2))$$
$$+\tfrac{1}{2}(R_{22}^{'(2)}\gamma_{21} + R_{21}^{(2)}E(x|y_2) + R_{24}^{(2)'}E(\gamma_1|y_2))'R_{22}^{(3)^{-1}}R_{14}^{(3)}R_{22}^{(3)^{-1}}(R_{22}^{'(2)}\gamma_{21}$$
$$+ R_{21}^{(2)}E(x|y_2) + R_{24}^{(2)'}E(\gamma_1|y_2)) + \gamma_{21}'R_{15}^{(1)}x-(R_{22}^{'(2)}\gamma_{21}$$
$$+ R_{21}^{(2)}E(x|y_2) + R_{24}^{(2)'}E(\gamma_1|y_2))'\}.$$

The new natural lower bound for the leader (J_1^{*}) is the infimum of $\tilde{J}_1$ over the product set $\Gamma_1 \times \Gamma_{21}$, i.e., $J_1^{*} \geq \inf\limits_{\gamma_1\in\Gamma_1,\gamma_{21}\in\Gamma_{21}} \tilde{J}_1(\gamma_{21},\gamma_{11},E(\gamma_1|y_2))$

Before we state the theorem which gives the optimal solution which achieves the tight lower bound of $\tilde{J}_1$, let us assume the following:

1. $R_{14}^{(1)} + R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)} R_{22}^{(3)^{-1}} R_{22}^{\prime(2)} - R_{14}^{(2)} R_{22}^{(3)^{-1}} R_{22}^{(2)\prime} - R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(2)\prime}$ is positive definite.
2. $R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)} R_{22}^{(3)^{-1}} R_{24}^{(2)\prime} - R_{12}^{(2)\prime} R_{22}^{(3)^{-1}} R_{24}' - R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{12}'$ is positive semi-definite.
3. $\bar{\lambda}[R_{11}^{-1}(R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)} R_{22}^{(3)^{-1}} R_{24}^{(2)\prime} - R_{12}^{(2)\prime} R_{22}^{(3)^{-1}} R_{24}' - R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{12}^{(2)\prime})] < 1$
   where $\bar{\lambda}(A)$ is as defined in the previous section.

Theorem 6.2: If assumptions 1, 2, and 3 are satisfied, then there exists a unique optimal team pair $(\gamma_1^t, \gamma_{21}^t)$ for the modified objective function $\tilde{J}_1$, where γ_1^t and γ_{21}^t can be expressed respectively, as

$$\gamma_1^t = h_o u_{21} + h_1 y_1 + h_2 y_2$$

$$\gamma_2^t = \bar{h} y_2.$$

Proof: $\tilde{J}_1$ is quadratic and under assumption 1, it is strictly convex in γ_{21}, so the optimal γ_{21} can be obtained by person-by-person optimization, and by taking the gradient of $\tilde{J}_1$ with respect to γ_{21} and setting it equal to zero. The optimal γ_{21} is

$$\gamma_{21}^* = kE(\gamma_1|y_2) + \bar{K}E(x|y_2)$$

where

$$k = (R_{14}^{(1)} + R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)} R_{22}^{(3)^{-1}} R_{22}^{(2)\prime} - R_{14}^{(2)} R_{22}^{(3)^{-1}} R_{22}^{(2)\prime}$$
$$- R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(2)\prime})^{-1} (R_{14}^{(1)\prime} - R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{12}^{(2)^{-1}} + R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{24}^{(2)\prime})$$

$$\bar{K} = (R_{14}^{(1)} + R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)} R_{22}^{(3)^{-1}} R_{22}^{(2)\prime} - R_{14}^{(2)} R_{22}^{(3)^{-1}} R_{22}^{(2)\prime}$$
$$- R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)\prime})^{-1} (R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{21}^{(2)} + R_{15}^{(1)} - R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{15}^{(2)}).$$

To find the min $\tilde{J}_1$ with respect to γ_1, we determine the first Frechet variation $\delta\tilde{J}_1$ of $\tilde{J}_1$, and set it equal to zero. After some lengthy manipulation, in which we use some properties of the conditional expectation, we get

$$R_{11}\gamma_1 + \bar{k}_2 u_{21} + \bar{K}_2 E(x|y_2) + k_3 E(\gamma_1|y_2) + R_{13}E(x|y_1,y_2) = 0$$

where

$$\bar{k}_2 = R_{12}^{(1)} - R_{12}^{(2)} R_{22}^{(3)^{-1}} R_{22}^{(2)'} - R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(2)'} + R_{24}^{(2)} R_{22}^{(3)'} R_{14}^{(3)'} R_{22}^{(3)^{-1}} R_{22}^{(2)'}$$

$$\bar{K}_2 = -R_{12}^{(2)} R_{22}^{(3)^{-1}} R_{21}^{(2)} + R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)} R_{22}^{(3)^{-1}} R_{21}^{(2)} - R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{15}^{(2)}$$

$$k_3 = -R_{12}^{(2)} R_{22}^{(3)^{-1}} R_{24}^{(2)'} - R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{12}^{(2)'} + R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{14}^{(3)} R_{22}^{(3)^{-1}} R_{24}^{(2)'} .$$

Assumption 2 guarantees that the second Frechet variation $\delta^2 J$ is positive semi-definite. The remainder of the proof is equivalent to the one given in [49], which uses assumption 3 and the Gaussian distribution properties to show the result. ∎

6.3.2. The Enforcement of the Modified Team Solution

Let the leader adopt a strategy which is of the same form as the one described in the previous section, i.e.

$$\gamma_1 = \gamma_1^t + A(u_{21} - \bar{h}y_2) = h_o u_{21} + h_1 y_1 + h_2 y_2 + A(u_{21} - \bar{h}y_2) .$$

Substituting for γ_1 from above and for γ_{22} from (6.3) in J_2 and taking the gradient of J_2 with respect to u_{21} and set it equal to zero, we get the following as the optimal response

$$
\begin{aligned}
E[\{ & (R_{21}^{(1)}-R_{22}^{(2)}R_{22}^{(3)^{-1}}R_{21}^{(2)})x+(R_{22}^{(1)}-R_{22}^{(2)}R_{22}^{(3)^{-1}}R_{22}^{(2)'})u_{21}+[(h_o+A)'(R_{24}^{(1)} \\
& -R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{22}^{'(2)}-R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}(h_o+A))+(R_{24}^{'(1)}-R_{22}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{'(2)} \\
& -(h_o+A)'R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'})(h_o+A)]u_{21}+(h_o+A)'(R_{23}x \\
& -R_{21}^{(2)}R_{22}^{(3)^{-1}}R_{21}^{(2)}E(x|y_2)-R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}h_1H_1E(x|y_2) \\
& -R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}(h_2-A\bar{h})y_2)+(R_{24}^{(1)}-R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{22}^{(2)'} \\
& -R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}h_1H_1E(x|y_2)-R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}(h_2-A\bar{h})y_2) \\
& +(R_{24}^{(1)}-R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{22}^{(2)'}-R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}(h_o+A))'(h_1H_1E(x|y_2) \\
& +(h_2-A\bar{h})y_2)+(h_o+A)'R_{24}^{(2)'}R_{22}^{(2)^{-1}}R_{24}^{(2)'}(h_o+A)u_{21}+(h_o+A)'(-R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{21}x \\
& +\frac{1}{2}R_{24}^{(2)}R_{22}^{(3)^{-1}}(R_{24}^{(2)'}h_1H_1+R_{21}^{(2)})E(x|y_2)+\frac{1}{2}R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}(h_2-A\bar{h})y_2) \\
& +\frac{1}{2}(h_o+A)'R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}(h_1'H_1E(x|y_2)+(h_2-A\bar{h})y_2) \\
& +\frac{1}{2}(h_o+A)'R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{21}^{(2)}E(x|y_2))\}|y_2]=0.
\end{aligned}
$$

Let $u_{21}=u_{21}^t=\bar{h}y_2$ in the above equation. After lengthy but straightforward manipulation we get the above equation equivalent to the following linear matrix equation

$$B_1A = B_2 \tag{6.4}$$

where

$$B_1 = \bar{h}'F_3'-\bar{h}'h_o'R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}+L'F_4'-h_2'R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}+h_1R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}$$

$$B_2 = -L_1'F_1'-\bar{h}'F_2'-\bar{h}'F_3'h_o+\bar{h}'h_o'R_{24}^{(2)}R_{22}^{(3)^{-1}}R_{24}^{(2)'}h_o-\bar{h}'h_o'F_3-L'F_4'h_o+h_2'R_{24}^{(2)}R_{22}^{(3)^{-1}}$$

$$R_{24}^{(2)'}h_o - L'H_1'h_1'F_3 - h_2'F_3 - L'H_1'h_1 R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{24}^{(2)'} h_o$$

and

$$F_1 = R_{21}^{(1)} - R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{21}^{(2)} \quad , \; F_2 = R_{22}^{(1)} - R_{22}^{(2)} R_{22}^{(3)^{-1}} R_{22}^{(2)'}$$

$$F_3 = R_{24}^{(1)} - R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{22}^{(2)'} \quad , \; F_4 = R_{23} - R_{21}^{(2)} R_{22}^{(3)^{-1}} R_{21}^{(2)} - R_{24}^{(2)} R_{22}^{(3)^{-1}} R_{24}^{(2)'} h_1 H_1$$

$$L = \Sigma H_2'(H_2 \Sigma H_2' + \Lambda_2)^{-1}.$$

The following theorem summarizes the results we obtained.

Theorem 6.3: If there exists at least one matrix A which satisfies the linear matrix equation $B_1 A = B_2$, and if $\gamma_1 \in \Gamma_1$ is picked as

$$\gamma_1 = \gamma_1^t + A(u_{21} - \gamma_{21}^t),$$

then the modified team solution is achieved.

The following example illustrates some of the basic ideas presented in this section.

Example: Let the objective functions of the leader and of the follower be as follows

$$J_1 = E\left\{\frac{1}{2} u_1^2 + u_1 [1 \quad 0] \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} + 2u_1 x + \frac{1}{2} [u_{21} \quad u_{22}] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} \right.$$

$$\left. + 4 [u_{21} \quad u_{22}] \begin{bmatrix} 1 \\ 0 \end{bmatrix} x \right\}$$

$$J_2 = E\left\{ [u_{21} \quad u_{22}] \begin{bmatrix} 0 \\ 1 \end{bmatrix} x + \frac{1}{2} [u_{21} \quad u_{22}] \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} + u_1 x + u_1 [1 \quad \tfrac{1}{3}] \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} \right\}$$

For simplicity, let

$$y = y_1 = y_2 = x \quad , \text{ and } \quad y_o = u_{21}$$

where x is a scalar random variable with $x \sim N(0,\sigma)$. If we minimize J_2 with respect to u_{22}, we find that

$$\gamma_{22}^*(y) = y + \frac{1}{8} E(\gamma_1|y).$$

If we substitute for u_{22} in J_1, and find the infimum of J_1 with respect to γ_{21} and γ_1, the result will be

$$\gamma_1^* = -u_{21} - \frac{17}{8} y - \frac{1}{64} E(\gamma_1|y)$$

$$\gamma_{21}^* = E(\gamma_1|y) + 4y.$$

Solving the above three equations, we get the modified team solution as

$$\gamma_1^* = -0.9846u_{21} - 2.092y$$

$$\gamma_{21}^* = 0.961y$$

$$\gamma_{22}^* = 0.6202y.$$

It can be easily checked that if the leader chooses his strategy of the form

$$\gamma_1 = -0.9846u_{21} - 2.092y + a(u_{21}-0.961y),$$

where $a = 4.166$, then he can enforce the above modified team solution, and the value of his objective function (J_1^*) will be

$$J_1^* = 0.11727\sigma.$$

For this example, J_1 is not strictly convex in the decision variables, so the global minimum of J_1 does not exist. J_1^* is the tight lower bound for J_1 which the leader can achieve, i.e., values of J_1 which are less than J_1^* cannot be induced.

CHAPTER 7

CONCLUSIONS

In this thesis, the role of information structure in some Nash and LF games is considered. We show that by preserving the information structure of the full order singularly perturbed LF games, reduced order solutions which are equivalent, in the limit as μ tends to zero, to the full order ones are obtained, but by using DDIS, solutions which are different from and more desirable than the NIS solutions are obtained. For example, in LF games, by using DDIS, the leader can, under certain conditions, achieve his most desirable solutions, which normally he cannot achieve. We investigate several classes of Nash and LF games with DDIS.

By preserving the information structure of the full order problems, while solving the reduced order ones, we designed, in Chapters 2 and 3, two well-posed methods to obtain reduced order and near optimal strategies for both linear closed loop and team LF games.

In Chapter 4, a class of two-person Nash games with DDIS is considered. Necessary conditions for existence of a Nash equilibrium solution with DDIS is derived for a single stage general duopoly model of a market structure. The case of linear market demand function and quadratic cost function is analyzed in detail and it is shown that the profit of the firm with DDIS is more than its corresponding profit with NIS, but the profit of the other firm is decreased compared to its profit in NIS. We extended our analysis of the concept of DDIS to multistage dynamic games. A two-stage duopoly game with DDIS is examined and sufficient conditions for the existence of an equilibrium solution of discrete linear quadratic Nash games with DDIS are given.

In Chapter 5, we showed the significance of using decision-dependent strategies by the leader in forcing cooperation of the Nash followers in a LF game. We gave a static model of a market which consists of two firms and the government. The two firms behave as Nash duopolists, while the government behaves as a coordinator. If the government adopts a certain representation for its strategy, which is affine in the output rates of the two firms, then it can always force the two firms to cooperate. We analyzed in detail the case of a linear demand relation and quadratic cost functions; we found explicit solutions for the optimal strategies of the firms and the government; and we compared the voluntary Pareto-optimal situation with the enforced Pareto-optimal one.

In Chapter 6, we solved two stochastic static LF team problems, where each player has a quadratic cost function and the random variables are normally distributed. The first problem is a 3-person stochastic optimal coordination. We showed that under a certain condition, the coordinator by adopting strategies which are linear in the decisions of the Nash followers can enforce cooperation. The second problem is a 2-person LF team game, in which the leader does not completely detect the decision variable of the follower. If the complete detectability condition is not satisfied, then the leader cannot enforce his global optimal solution. We defined a new modified team problem in which we took into account the optimal response of the undetected action of the follower. We found out that the leader can, under a certain condition, achieve this new tight lower bound.

In the area of information structure in Nash and LF strategies, there are several avenues which have yet to be explored, such as

1. The sensitivity of Nash and LF equilibrium solutions to both uncertainties and changes in the information structure. In particular, we may try to develop robust strategies (strategies which are insensitive to uncertainties and changes in the information structure), which the leader can adopt to achieve his team solution.
2. The study of information structure design, i.e. who should know what. We are still at a very elementary stage and several difficulties have to be overcome, before we are able to answer the above question in a unified and systematic manner, such as, a) A deeper understanding of the subject of dynamic information structure, b) more investigation on the matter of incentives and decision dependent information structure.
3. Investigation of the generality of the effectiveness of preserving the information structure in obtaining near optimal reduced order strategies and well-posedness. For example, we can check whether preserving the information structure of the full order problem, while solving the reduced order ones, leads to well-posed solutions for some classes of nonlinear game problems.

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APPENDIX A

NECESSARY CONDITIONS FOR THE LEADER'S MINIMIZATION PROBLEM

Applying the matrix minimum principle [14] to the leader optimization problem we get the following set of matrix algebraic equations:

$$P_1A'_{11} - P_1K_{11}\bar{S}_{11} - P_1L'_{21}B'_{12} - P_1K_{12}\bar{S}_{13}\pi'_1 - P_1K_{12}\bar{S}'_{12} + P_1\tilde{A}'_{21}\pi'_1 + P_2K_2\pi_2$$

$$+P_2K_{12}\bar{S}'_{22} + P_2K_{12}\bar{S}_{23}\pi'_1 + P_2K_{11}\bar{S}_{21} + P_2K_{11}\bar{S}_{22}\pi'_1 + P_2\tilde{Q}_{22}\hat{A}^{-1}_{22}(\bar{S}'_{12} + \bar{S}_{13}\pi'_1) -$$

$$P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})' \cdot (K_{13}\bar{S}'_{22} + K_{13}\bar{S}_{23}\pi'_1) - P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})'\tilde{Q}_{23}\hat{A}^{-1}_{22}(\bar{S}'_{12} + \bar{S}_{13}\pi'_1)$$

$$+A_{11}P_1 - \bar{S}_{11}K_{11}P_1 - B_{12}L_{21}P_1 - \pi_1\bar{S}_{13}K'_{12}P_1 - \bar{S}_{12}K_{12}P_1 + \pi_1\tilde{A}_{21}P_1 +$$

$$\pi'_2K_2P_2 + \pi_1\bar{S}'_{22}K_{11}P_2 + \pi_1\bar{S}_{23}K'_{12}P_2 + \bar{S}_{21}K_{11}P_2 + \bar{S}_{22}K'_{12}P_2$$

$$- (\pi_1\bar{S}_{23}K_{13} + \bar{S}_{22}K_{13})\hat{A}^{-1}_{22}\hat{A}_{21}P_2 + (\pi_1\bar{S}_{13} + \bar{S}_{12}) \cdot (\hat{A}^{-1}_{22})'\tilde{Q}'_{22}P_2 - (\bar{S}_{12} + \pi_1\bar{S}_{13})(\hat{A}^{-1}_{22})'\tilde{Q}_{23}$$

$$(\hat{A}^{-1}_{22})\hat{A}_{21}P_2 = 0 \tag{A.1}$$

$$P_2A'_0 + A_0P_2 + I = 0 \tag{A.2}$$

$$-B'_{12}K_{11}P_1 - \pi_3\bar{S}_{13}K'_{12}P_1 + \pi_3\tilde{A}_{21}P_1 - B'_{22}K'_{12}P_1 + \pi'_4K_2P_2 + R_{22}L_{21}P_2$$

$$+\pi_3\bar{S}_{23}K'_{12}P_2 + \pi_3\bar{S}'_{22}K_{11}P_2 + R_{12}L_{21}P_1 - (R_{22}L_{22} + \pi_3\bar{S}_{23}K_{13})\hat{A}^{-1}_{22}\hat{A}_{21}P_2 +$$

$$(\pi_3\bar{S}_{13} + B'_{22})(\hat{A}^{-1}_{22})'\tilde{Q}'_{22}P_2 - (B'_{22} + \pi_3\bar{S}_{13})(\hat{A}^{-1}_{22})'\tilde{Q}_{23}\hat{A}^{-1}_{22}\hat{A}_{21}P_2 = 0 \tag{A.3}$$

$$-\pi'_5P_1K_{12}\bar{S}_{13}(\hat{A}^{-1}_{22})' + \pi'_5P_1\tilde{A}'_{21}(\hat{A}^{-1}_{22})' + \pi'_5P_2K_2\pi_6 + \bar{B}'_2K_2P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})' + \pi'_5P_2K_{11}\bar{S}_{22}(\hat{A}^{-1}_{22})'$$

$$+\pi'_5P_2K_{12}\bar{S}_{23}(\hat{A}^{-1}_{22})' - R_{22}L_{21}P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})' - \pi'_5P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})'K_{13}\bar{S}_{23}(\hat{A}^{-1}_{22})' - B'_{22}(\hat{A}^{-1}_{22})'\tilde{Q}'_{22}$$

$$P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})' + \pi'_5P_2\tilde{Q}_{22}\hat{A}^{-1}_{22}\bar{S}_{13}(\hat{A}^{-1}_{22})' - \pi'_5P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})'\tilde{Q}'_{23}(\hat{A}^{-1}_{22})\bar{S}_{13}(\hat{A}^{-1}_{22})'$$

$$+B'_{22}(\hat{A}^{-1}_{22})'\tilde{Q}_{23}(\hat{A}^{-1}_{22})\hat{A}_{21}P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})' + R_{22}L_{22}(\hat{A}^{-1}_{22})\hat{A}_{21}P_2\hat{A}'_{21}(\hat{A}^{-1}_{22})' -$$

$$[R_{12}L_{22} - B'_{22}K_{13}]P_3 = 0 \tag{A.4}$$

$$-\hat{A}_{22}^{-1}\tilde{A}_{21}P_1\hat{A}_{21}' + \hat{A}_{22}^{-1}\bar{S}_{13}K_{12}'P_1\hat{A}_{21}' + \hat{A}_{21}P_2K_2'(\bar{S}_{12} - \hat{A}_{12}\hat{A}_{22}^{-1}\bar{S}_{13})(\hat{A}_{22}^{-1})'$$

$$+\hat{A}_{22}^{-1}\hat{A}_{21}P_2K_2(\bar{S}_{12} - \hat{A}_{12}\hat{A}_{22}^{-1}\bar{S}_{13}) - \hat{A}_{22}^{-1}\hat{A}_{21}P_2\tilde{Q}_{22}\hat{A}_{22}^{-1}\bar{S}_{13} - \hat{A}_{21}P_2\tilde{Q}_{22}\hat{A}_{22}^{-1}\bar{S}_{13}(\hat{A}_{22}^{-1})'$$

$$-\hat{A}_{22}^{-1}\hat{A}_{21}P_2K_{11}\bar{S}_{22} - \hat{A}_{22}^{-1}\hat{A}_{21}P_2K_{12}\bar{S}_{23} + \hat{A}_{21}P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})'K_{13}\bar{S}_{23}(\hat{A}_{22}^{-1})' +$$

$$\hat{A}_{21}P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{23}\hat{A}_{22}^{-1}\bar{S}_{13}(\hat{A}_{22}^{-1})' + (\hat{A}_{22}^{-1})\hat{A}_{21}P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})'\tilde{Q}_{23}\hat{A}_{22}^{-1}\bar{S}_{13} + (\hat{A}_{22}^{-1})\hat{A}_{21}P_2$$

$$\hat{A}_{21}'(\hat{A}_{22}^{-1})'K_{13}\bar{S}_{23} + \hat{A}_{22}P_3 - (\hat{A}_{22}^{-1})\bar{S}_{22}'K_{11}P_2\hat{A}_{21}' - \hat{A}_{22}^{-1}\bar{S}_{23}K_{12}'P_2\hat{A}_{21}' - \hat{A}_{21}P_1\tilde{A}_{21}'(\hat{A}_{22}^{-1})'$$

$$+\hat{A}_{21}P_1K_{12}(\hat{A}_{22}^{-1})' + (\hat{A}_{22}^{-1})(\bar{S}_{12} - \hat{A}_{12}\hat{A}_{22}^{-1}\bar{S}_{13})'K_2P_2\hat{A}_{21}' + (\bar{S}_{12} - \hat{A}_{12}\hat{A}_{22}^{-1}\bar{S}_{13})'K_2P_2\hat{A}_{21}'$$

$$(\hat{A}_{22}^{-1})' - \bar{S}_{13}(\hat{A}_{22}^{-1})'\tilde{Q}_{22}'P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})' - (\hat{A}_{22}^{-1})\bar{S}_{13}(\hat{A}_{22}^{-1})'\tilde{Q}_{22}'P_2\hat{A}_{21}' - \bar{S}_{22}'K_{11}P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})'$$

$$-\bar{S}_{23}K_{12}'P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})' + (\hat{A}_{22}^{-1})\bar{S}_{23}K_{13}(\hat{A}_{22}^{-1})\hat{A}_{21}P_2\hat{A}_{21}' + (\hat{A}_{22}^{-1})\bar{S}_{13}(\hat{A}_{22}^{-1})'\tilde{Q}_{23}(\hat{A}_{22}^{-1})$$

$$\hat{A}_{21}P_2\hat{A}_{21}' + \bar{S}_{13}(\hat{A}_{22}^{-1})'\tilde{Q}_{23}(\hat{A}_{22}^{-1})\hat{A}_{21}P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})' + \bar{S}_{23}K_{13}(\hat{A}_{22}^{-1})\hat{A}_{21}P_2\hat{A}_{21}'(\hat{A}_{22}^{-1})'$$

$$+P_3\hat{A}_{22}' - \hat{A}_{21}P_2K_{11}\bar{S}_{22}(\hat{A}_{22}^{-1})' - \hat{A}_{21}P_2K_{12}\bar{S}_{23}(\hat{A}_{22}^{-1})' = 0 \qquad (A.5)$$

where

$$\pi_1 = (-A_{12} + B_{12}L_{22} + \bar{S}_{12}K_{13})\hat{A}_{22}^{-1}$$

$$\pi_2 = -\bar{S}_{11} - \bar{S}_{12}\pi_1' + \hat{A}_{12}\hat{A}_{22}^{-1}(\bar{S}_{12}' + \bar{S}_{13}\pi_1')$$

$$\pi_3 = (-R_{12}L_{22} + B_{22}'K_{13})\hat{A}_{22}^{-1}$$

$$\pi_4 = [-\bar{S}_{12}\pi_3' - B_{12} + \hat{A}_{12}\hat{A}_{22}^{-1}(B_{22} + \bar{S}_{13}\pi_3')]$$

$$\pi_5 = -[L_{21}'R_{12} - K_{11}B_{12} + (Q_{12} + L_{21}'R_{12}L_{22} - K_{11}A_{12} - K_{11}B_{12}L_{22}$$
$$+ \tilde{A}_{21}'K_{13})\hat{A}_{22}^{-1}B_{22}]$$

$$\Pi_6 = -\bar{S}_{12}(\hat{A}_{22}^{-1})' + \hat{A}_{12}\hat{A}_{22}^{-1}\bar{S}_{13}(\hat{A}_{22}^{-1})' \ , \ \tilde{B}_2 = B_{12} - \hat{A}_{12}\hat{A}_{22}^{-1}B_{22}$$

So in order for the leader to find L_{21}, L_{22} he has to solve equations (2.3), (2.6), (2.7), (2.8), and (A.1) through (A.5).

APPENDIX B. DECOMPOSITION OF THE FULL ORDER LEADER-FOLLOWER TEAM PROBLEM

Equations (3.6) - (3.11) of the full order system is decomposed and the limit as μ tends to zero is taken as follows:

Substituting the forms of $K(t,\mu), Q_i$ as given before in equation (3.6) and letting $\mu \to 0^+$, we get

$$\dot{K}_1(t,0) + K_1(t,0)A_{11}(t) + K_2(t,0)A_{21}(t) + A_{11}'(t)K_1(t,0) + A_{21}'(t)K_2'(t,0) + Q_{11}(t) - K_1(t,0)\bar{S}_1K_1(t,0) - K_2(t,0)\bar{S}_2'K_1(t,0) - K_1(t,0)\bar{S}_2K_2'(t,0) - K_2(t,0)\bar{S}_3K_2'(t,0) = 0 \qquad (B.1)$$

$$K_1(t,0)A_{12}(t) + K_2(t,0)A_{22}(t) + A_{21}'(t)K_3(t,0) + Q_{12}(t) - K_1(t,0)\bar{S}_2K_3(t,0) - K_2(t,0)\bar{S}_3K_3(t,0) = 0 \qquad (B.2)$$

$$K_3(t,0)A_{22}(t) + A_{22}'(t)K_3(t,0) - K_3(t,0)\bar{S}_3K_3(t,0) + Q_{13}(t) = 0. \qquad (B.3)$$

By letting $\mu \to 0^+$, in equation (8) we have

$$R_{22}^{-1}(t)[B_{12}'P_1(t,0) + B_{22}'P_3(t,0)] = R_{12}^{-1}(t)[B_{12}'K_1(t,0) + B_{22}'K_2'(t,0)] \qquad (B.4)$$

$$R_{22}^{-1}(t)B_{22}'P_4(t,0) = R_{12}^{-1}(t)B_{22}'K_3(t,0). \qquad (B.5)$$

Equation (3.10) will be after letting $\mu \to 0^+$

$$F_1(t,0) = R_{21}R_{11}^{-1}(B_{11}'K_1(t,0) + B_{21}'K_2'(t,0)) - B_{11}'P_1(t,0) - B_{21}'P_3(t,0) \qquad (B.6)$$

$$F_2(t,0) = R_{21}R_{11}^{-1}B_{21}'K_3(t,0) - B_{21}'P_4(t,0). \qquad (B.7)$$

Decomposing equation (3.11), where the gains of the controls are evaluated at $\mu=0$

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix} A_{11}-S_{11}K_1(t,0)-S_{12}K_2'(t,0)-S_{21}K_1(t,0)-S_{22}K_2'(t,0) & A_{12}-S_{12}K_3(t,0)-S_{23}K_3(t,0) \\ \dfrac{A_{21}-S_{12}'K_1(t,0)-S_{13}K_2'(t,0)-S_{22}'K_1(t,0)-S_{23}K_2'(t,0)}{\mu} & (A_{22})-S_{13}K_3(t,0)-S_{23}K_3(t,0))/\mu \end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}. \quad (B.8)$$

Let

$$\begin{bmatrix}x\\ z\end{bmatrix} = L\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

where

$$L = \begin{bmatrix} I_1 & -\mu S \\ -T & I_2+\mu TS\end{bmatrix}; \quad L^{-1} = \begin{bmatrix} I_1+\mu TS & +\mu S \\ T & I_2\end{bmatrix}$$

and

$$\mu\dot{T} = \tilde{A}_{22}T - \mu T\tilde{A}_{11} + \mu T\tilde{A}_{12}T - \tilde{A}_{21}$$

$$\mu\dot{S} = -\mu[\tilde{A}_{11}-\tilde{A}_{12}T(t,\mu)]S - S[\tilde{A}_{22} + \mu T(t,\mu)\tilde{A}_{12}] - \tilde{A}_{12}.$$

Substituting for x,z in terms of v_1, v_2 in (B.8), we get

$$\dot{v}_1 = (\tilde{A}_{11}-\tilde{A}_{12}T(t,\mu))v_1 \quad (B.9)$$

$$\mu\dot{v}_2 = (\tilde{A}_{22} + \mu T(t,\mu)\tilde{A}_{12})v_2 \quad (B.10)$$

Let $\Psi_{11}(t,t_o,\mu)$, $\Psi_{22}(t,t_o,\mu)$ be the state transition matrices of (B.9) and (B.10) respectively, then they satisfy the following equations

$$\dot{\Psi}_{11}(t,t_o,\mu) = (\tilde{A}_{11}-\tilde{A}_{12}T(t,\mu))\Psi_{11}(t,t_o,\mu) \qquad \Psi_{11}(t_o,t_o,\mu) = I \quad (B.11)$$

$$\dot{\Psi}_{22}(t,t_o,\mu) = (\tilde{A}_{22} + \mu T(t,\mu)\tilde{A}_{12})\Psi_{22}(t,t_o,\mu) \qquad \Psi_{22}(t_o,t_o,\mu) = I \quad (B.12)$$

where

$$\tilde{A}_{11} = A_{11} - S_{11}K_1(t,0) - S_{12}K_2'(t,0) - S_{21}K_1(t,0) - S_{22}K_2'(t,0)$$

$$\tilde{A}_{12} = A_{12} - S_{13}K_3(t,0) - S_{23}K_3(t,0)$$

$$\tilde{A}_{21} = A_{21} - S_{12}'K_1(t,0) - S_{22}'K_1(t,0) - S_{23}K_2'(t,0) - S_{13}K_2'(t,0)$$

$$\tilde{A}_{22} = A_{22} - S_{13}K_3(t,0) - S_{23}K_3(t,0)$$

$$S_{11} = B_{11}R_{11}^{-1}B_{11}'; \quad S_{12} = B_{11}R_{11}^{-1}B_{21}'; \quad S_{13} = B_{21}R_{11}^{-1}B_{21}'$$

$$S_{21} = B_{12}R_{12}^{-1}B_{12}'; \quad S_{22} = B_{12}R_{12}^{-1}B_{22}'; \quad S_{23} = B_{22}R_{12}^{-1}B_{22}'.$$

From [23], we have

$$\lim_{\mu\to 0^+} T(t,\mu) = \tilde{A}_{22}^{-1}\tilde{A}_{21}$$

$$\lim_{\mu\to 0^+} S(t,\mu) = -\tilde{A}_{12}\tilde{A}_{22}^{-1}.$$

Substituting for $\begin{bmatrix} x \\ z \end{bmatrix} = L\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in (3.2), and then letting $\mu\to 0^+$, we have

$$\int_{t_o}^{t} (d_s\eta_1(t,s,0) - d_s\eta_2(t,s,0)\tilde{A}_{22}^{-1}\tilde{A}_{21})\Psi_{11}(s,t,0) = -R_{11}^{-1}[B_{11}'K_1(t,0) + B_{21}'K_2'(t,0)] + R_{11}^{-1}B_{21}'K_3(t,0)\tilde{A}_{22}^{-1}\tilde{A}_{21} \tag{B.13}$$

$$\int_{t_o}^{t} d_s\eta_2(t,s,0)\Psi_{22}(s,t,0) = -R_{11}^{-1}B_{21}'K_3(t,0). \tag{B.14}$$

Using the transformation $y = L\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in (3.4), and letting $\mu\to 0^+$, we get

$$P_1(t,0) - \int_{t}^{t_f} [(Q_{21}' + A_{11}'P_1(\tau,0) + A_{21}'P_3(\tau,0) + \eta_1'(\tau,t,0)F_1(\tau,0)) - (Q_{22} + A_{21}'P_4(\tau,0) + \eta_1'(\tau,t,0)F_2(\tau,0))\tilde{A}_{22}^{-1}\tilde{A}_{21}]\Psi_{11}(\tau,t,0)d\tau = 0 \tag{B.15}$$

$$Q_{22} + A_{21}'P_4(\tau,0) + \eta_1'(\tau,t,0)F_2(\tau,0) = 0 \qquad \forall t \leqslant \tau \tag{B.16}$$

$$Q_{22}' + A_{12}'P_1(\tau,0) + A_{22}'P_3(\tau,0) + \eta_2'(\tau,t,0)F_1(\tau,0) = 0 \quad \forall t \leqslant \tau \tag{B.17}$$

$$Q_{23} + A_{22}'P_4(\tau,0) + \eta_2'(\tau,t,0)F_2(\tau,0) = 0 \qquad \forall t \leqslant \tau \tag{B.18}$$

APPENDIX C. THE SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE FAST TEAM LEADER-FOLLOWER GAME

The sufficient conditions for the existence of an optimal leader-follower fast strategy pair which coincides with a team fast strategy pair are as follows:

If there exists a function $\eta_f(t,\theta)$ with $\eta_f(t,\theta)=0$ for $\theta > t$ and $n_2\times n_2$ matrix K_{2f} which satisfy

$$\int_{t_o}^{t} d_s\eta_f(t,s)\phi_f(s,t) = -R_{11}^{-1}B_{21}'K_{1f} \tag{C.1}$$

$$R_{22}^{-1}B_{22}'K_{2f} = R_{12}^{-1}B_{22}'K_{1f} \tag{C.2}$$

$$\mu K_{2f}(t) - \int_{t}^{t_f} (Q_{23}(\tau) + A_{22}'(\tau)K_{2f}(\tau) + \eta_f'(\tau,t)F_f(\tau))\phi_f(\tau,t)d\tau = 0 \tag{C.3}$$

$$\mu\dot{K}_{1f} + K_{1f}A_{22} + A_{22}'K_{1f} + Q_{13} - K_{1f}(B_{21}R_{11}^{-1}B_{21}' + B_{22}R_{12}^{-1}B_{22}')K_{1f} = 0 \tag{C.4}$$

$$K_{1f}(t_f) = 0$$

where

$$F_f(t) = R_{21}R_{11}^{-1}B_{21}'K_{1f} - B_{21}'K_{2f} \tag{C.5}$$

$$\mu\dot{\phi}_f(t,t_o) = (A_{22} - B_{21}R_{11}^{-1}B_{21}'K_{1f} - B_{22}R_{22}^{-1}B_{22}'K_{2f})\phi_f(t,t_o) \tag{C.6}$$

$$\phi_f(t_o,t_o) = I$$

then

$$u_{1f}^* = \int_{t_o}^{t} d_s\eta_f(t,s)z_f(s)$$

$$u_{2f}^* = -R_{12}^{-1}B_{22}'K_{1f}z_f(t)$$

is the optimal fast leader-follower strategy pair. Furthermore, this is an optimal team solution.

APPENDIX D. THE SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE HYBRID SLOW TEAM LEADER-FOLLOWER GAME

The sufficient conditions for the existence of an optimal leader-follower team hybrid slow strategy pair are as follows. If there exists a function $\eta_s(t,\theta)$ with $\eta_s(t,\theta)=0$ for $\theta > t$ and $n_1 \times n_1$ matrix P which satisfy

$$\int_{t_o}^{t} [d_s\eta_s(t,s)-d_s\eta_f(t,s,0)A_{22}^{-1}(A_{21}-B_{21}M_{1s}-B_{22}M_{2s})]\phi_s(s,t) = -M_{1s} \tag{D.1}$$

$$\tilde{Q}_{22}'(\tau)-\bar{Q}_{23}(\tau)M_{1s}(\tau)+\hat{B}_{12}'(\tau)P_s(\tau)-B_{22}'(\tau)(A_{22}^{-1})'\eta_f'(\tau,t,0)F_s(\tau)-\tilde{R}_{22}M_{2s}=0 \quad \text{for } t<\tau \tag{D.2}$$

$$P_s(t)-\int_t^{t_f}[\tilde{Q}_{21}'-\tilde{Q}_{22}M_{2s}-\bar{Q}_{22}M_{1s}+\hat{A}_{11}'P_s(\tau)+(\eta_s'(\tau,t) -(A_{22}^{-1}A_{21})'\eta_f'(\tau,t,0))F_s(\tau)]\phi_s(\tau,t)d\tau = 0 \qquad \forall t<\tau \tag{D.3}$$

where

$$\bar{Q}_{22}'(\tau)-\tilde{R}_{21}M_{1s}(\tau)+\tilde{B}_{11}'P_s(\tau)+\dot{F}_s(\tau)-B_{21}'(A_{22}^{-1})'\eta_f(\tau,t,0)F_s(\tau)-\bar{Q}_{23}'M_{2s}(\tau)=0 \tag{D.4}$$

and

$$\dot{\phi}_s(t,t_o) = (\hat{A}_{11}-\hat{B}_{11}M_{1s}-\hat{B}_{12}M_{2s})\phi_s(t,t_o) \tag{D.5}$$

$$\phi_s(t_o,t_o) = I$$

then

$$\tilde{u}_{1s}(t) = \int_{t_o}^{t}(d_s\eta_s(t,s)-d_s\eta_f(t,s)A_{22}^{-1}A_{21})x_s(s)-\int_{t_o}^{t}d_s\eta_f(t,s)A_{22}^{-1}B_{21}u_{1s}(s) -\int_{t_o}^{t}d_s\eta_f(t,s)A_{22}^{-1}B_{22}u_{2s}$$

$$u_{2s} = -M_{2s}x_s(t)$$

constitute an optimal leader-follower strategy pair which coincides with the team solution.

APPENDIX E. PROOF OF LEMMA 3.4

Define $\Psi_o = L^{-1}\varphi_o$ where

$$L = \begin{bmatrix} I_1 & -\mu S \\ -T & I_2+\mu TS \end{bmatrix} \quad ; \quad L^{-1} = \begin{bmatrix} I_1+\mu TS & +\mu S \\ T & I_2 \end{bmatrix}$$

and T,S satisfies

$$\mu\dot{T} = A_{22}T - \mu TA_{11} + \mu TA_{12}T - A_{21}$$

$$\mu\dot{S} = -\mu[A_{11}-A_{12}T]S - S[A_{22}+\mu TA_{12}] - A_{12}$$

then $\psi_o(t,t_o,\mu)$ satisfies

$$\dot{\psi}_o(t,t_o,\mu) = \begin{bmatrix} A_{11}-A_{12}T(t,\mu) & 0 \\ 0 & \dfrac{A_{22}+\mu TA_{12}}{\mu} \end{bmatrix} \psi_o(t,t_o,\mu).$$

In a proof similar to the one given in [22, p. 16], we can show that $T(t,\mu)$ is continuously differentiable and bounded for all $t > t_o$ and $\forall\mu \in [0,\mu^*)$ where μ^* is small positive parameter, and it satisfies

$$T(t,\mu) = A_{22}^{-1}A_{21} + 0(\mu).$$

To prove the lemma, it is sufficient to prove that $\psi_{01}(t,t_o,\mu)$ and $\psi_{02}(t,t_o,\mu)\,\dfrac{\bar{B}_2}{\mu}$ are bounded in the limit as μ tends to zero, where $\psi_{01}(t,t_o,\mu)$ and $\psi_{02}(t,t_o,\mu)$ satisfy the following

$$\dot{\psi}_{01}(t,t_o,\mu) = (A_{11}-A_{12}A_{22}^{-1}A_{21} + 0(\mu))\psi_{01}(t,t_o,\mu)$$

$$\psi_{02}(t,t_o,\mu) = \frac{A_{22}+0(\mu)}{\mu})\psi_{02}(t,t_o,\mu)$$

since $A_{11}-A_{12}A_{22}^{-1}A_{21}$ is bounded, so

$$\|\psi_{01}(t,t_o,0)\| < K_1 e^{\gamma_1|t-t_o|}$$

for some positive constants K, γ, (see [21, p. 287]). Also using a proof similar to the one in [22, p. 15], we can show

$$\|\psi_{02}(t,t_o,\mu)\| < K_2 e^{-\frac{\gamma_2}{\mu}(t-t_o)}$$

where K_2, γ_2 are positive constants. As a result,

$$\|\psi_{02}(t,t_o,\mu)\,\frac{\bar{B}_2}{\mu}\| < \frac{K_3}{\mu}\,e^{-\frac{\gamma_2}{\mu}(t-t_o)}$$

which is bounded at $\mu=0$ $\forall t > t_o$. ∎

VITA

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